~~NEW YORK UNIVERSITY~~
~~INSTITUTE OF [illegible]~~ [illegible] SCIENCES
~~LIBRARY~~
~~25 Waverly Place, New York 3, N. Y.~~

~~New York University~~
~~Institute of Mathematical~~ Mathematical Sciences
~~25 Waverly Place, New York 3, N.Y.~~

[illegible]

Moses, H. E.

Electrons and positrons in a time-independent electromagnetic field: A solution in the Schrödinger picture.

IMM-NYU 179

March 1952

284-2M-1053

IMM-NYU #179

Electrons and Positrons in a Time-Independent Electromagnetic Field: A Solution in the Schrödinger Picture.

By H. E. Moses

Prepared under
Navy Contract N6ori-201 Task Order No. 1
by the
Institute for Mathematics and Mechanics
New York University
March, 1952

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y

Electrons and Positrons in a Time-Independent Electromagnetic Field: A Solution in the Schrödinger Picture.

Part I

Introduction

The problem considered in the following is the solution of the Schrödinger equation for the state vector of an electron-positron field when this field is coupled to a time-independent external electromagnetic field. The method of solution used in the present discussion is completely analogous to the method used in solving the Schrödinger equation with a time independent Hamiltonian for the one electron problem.

In the one-electron problem one finds the spectral representation of the Hamiltonian of the problem. If the Hamiltonian can be considered as the sum of a Hamiltonian whose spectral representation is known and an additional term which may be considered small, then one can diagonalize the total Hamiltonian by means of a canonical transformation which when expressed in a suitable representation, satisfies an integral equation, that can be solved by a perturbation procedure.

In order to find the solution of the Schrödinger equation for the case of an electron-positron field coupled to an electromagnetic field, we proceed in a similar manner. We consider the Hamiltonian to consist of a part whose spectral representation is given (i.e. the Hamiltonian without interaction) and a small perturbation and we find the canonical transformation which diagonalizes the total Hamiltonian. The solution of the Schrödinger equations is obtained immediately.

Of course, the idea of finding a suitable canonical transformation is not a new one. However, it appears that the canonical transformation has not yet been treated in the number (or Fock) representation. It is possible to obtain simple results for the transformation in terms of this representation. It will be shown that in the number representation, the canonical transformation satisfies an integro-difference equation which can be solved by a perturbation scheme. Having found the canonical transformation, expectation values of various operators can be found as functions of time. Some results will be discussed.

The present paper is divided into two parts. In Part I, the number representation is set up for the electron-positron field and various operators are defined in this representation. Furthermore, the canonical transformation is obtained to the second order in the electron charge in terms of the one-particle canonical transformations and the expectation values of some operators are given in terms of the one-particle transformations.

In Part II, the explicit form of the one-particle transformations is used in the expressions for the expectation values, and the expectation values are given explicitly in terms of the external potentials. The results are then compared with the results obtained by other means. It is hoped that the program outlined above will clarify the source of some of the infinities occurring in field theory.

The Number Representation

In order to set up the number representation for the electron-positron field, it will be useful to review some of the properties of the Dirac Hamiltonian $\Omega^{(0)}$ for the one electron problem in the absence of an electromagnetic field. (By the expressions "one-electron problem" or "theory of the single electron" we mean the usual first quantization theory of relativistic electrons, unless otherwise noted.) The spectral representation of the Hamiltonian $\Omega^{(0)}$ can be expressed in terms of a complete set of variables consisting of the momentum vector operator P_i, the sign of the energy $E = \operatorname{sgn} \Omega^{(0)}$, and the component of the spin in the direction of motion T. This complete set of commuting operators will be denoted collectively by $S = \left\{ P_i, E, T \right\}$ with corresponding eigenvalues or quantum variables collectively denoted by $s = \left\{ p_i, \varepsilon, \tau, \right\}$. Here ε, τ each take on the values ± 1. The energy operator $\Omega^{(0)}$ is the usual field-free Dirac Hamiltonian

$$\Omega^{(0)} = - \sum_i a_i P_i - \beta m, \quad (\hbar = c = 1)$$

a, β being the usual Dirac operators. In terms of the quantum variables

$$\omega^2(s) = m^2 + p^2$$

where ω is the quantum variable associated with $\Omega^{(0)}$, $p^2 = \sum_{i=1}^{3} p_i^2$, and m is the mass of the electron.

Often it is useful to use another complete set of commuting operators to express the spectral representation of $\Omega^{(o)}$, this set consisting of the energy $\Omega^{(o)}$ itself, angular variables Θ, Φ representing the direction of motion, and the variable T introduced above. This set of operators $\{\Omega^{(o)}, \Theta, \Phi, T\}$ will also be denoted by S, since for much of what follows it does not matter which of the two sets of quantum variables is chosen. Here the quantum variables s consist of the set $\{\omega, \vartheta, \phi, \tau\}$ with $|\omega| \geq m$ $o \leq \vartheta \leq 2\pi$, $o < \phi < \pi$. It might be noted that the eigenfunctions of $\Omega^{(o)}$ given in most textbooks express the spectral representation of $\Omega^{(o)}$ in terms of the former set of quantum variables. However, for the calculation of transition probabilities, the latter set of quantum variables is more useful. There is a very simple relation between the representers of a state in both representations which can be obtained in an obvious way.

Having chosen our system of quantization variables we will set up our number representation. Following a suggestion of K. O. Friedrichs a state of the field Ψ is represented by a set of functions $\psi_n(s_1, s_2, \ldots, s_n) \equiv \psi_n(s)_n$ or

$$\Psi \underset{N}{\longleftrightarrow} \left\{\psi_n(s)_n\right\}$$

where the functions $\psi_n(s)_n$ are anti-symmetric in their

arguments $s_1, \ldots, s_n$. If all $\psi_n(s)_n$ vanish except for $n = m$ then the field consists exactly of m particles. It should be noted that the representation as introduced here does not distinguish between electrons and positrons and is thus symmetric in these entities, as opposed to the more conventional "hole" theory. The inner product of two states $\Psi^{(1)} \underset{N}{\longleftrightarrow} \left\{ \psi_n^{(1)}(s)_n \right\}$ and $\Psi \underset{N}{\longleftrightarrow} \left\{ \psi_n(s)_n \right\}$ is $\left(\Psi^{(1)}, \Psi \right) = \sum_n \int \bar{\psi}_n^{(1)}(s)_n \psi_n(s)_n (ds)_n$ where $(ds)_n$ means integration over the variables $s_1, s_2, \ldots s_n$.

The integration over the variables s is interpreted as a summation over the discrete variables and ordinary integration over the continuous variables.

We now introduce the annihilation and creation operators $A^-(s)$ and $A^+(s)$ which operate on the state vector $\Psi \underset{N}{\longleftrightarrow} \left\{ \psi_n(s)_n \right\}$ in the conventional manner.

$$(1) \qquad A^-(s')\Psi \underset{N}{\longleftrightarrow} \left\{ \sqrt{n+1}\ \psi_{n+1}(s)_n(s') \right\}$$

where $\psi_{n+1}(s)_n(s')$ means $\psi_{n+1}\ (s_1, \ldots, s_n, s')$.

Also

$$(2) \quad A^+(s')\Psi \underset{N}{\longleftrightarrow} \left\{ \sqrt{n}\ \underset{s_1 \ldots s_n}{\mathrm{Asy}}\ \psi_{n-1}(s)_{n-1} \delta(s_n, s') \right\}$$

where $\underset{s_1 \ldots s_n}{\mathrm{Asy}}\ f(s_1 \ldots s_n)$ means the asymmetric part of

$f(s_1 \ldots s_n)$, which is the sum of all terms

$\frac{(-1)^p f(s_{a_1}, s_{a_2} \ldots s_{a_n})}{n!}$ with $s_{a_1}, s_{a_2} \ldots s_{a_n}$ being a permutation of the variables $s_1, s_2, \ldots s_n$ and p being the number of interchanges required to effect this permutation.

The function $\delta(s,s')$ introduced in (2) is a generalized δ-function defined by

$$(3)\quad \begin{cases} \int_a f(s')\delta(s,s')ds' = f(s) & \text{if I includes s} \\ \qquad\qquad\qquad\qquad = 0 & \text{if I does not include s} \end{cases}$$

I being the range of integration.

As usual, $A^+(s)$ and $A^-(s)$ obey the commutation rules for fermion fields

$$(4)\quad \begin{aligned} \left\{A^+(s),\, A^-(s')\right\} &= A^+(s)\, A^-(s') + A^-(s')A^+(s) \\ &= \delta(s,s') \end{aligned}$$

$$(5)\quad \left\{A^+(s),\, A^+(s')\right\} = 0$$

$$(6)\quad \left\{A^-(s),\, A^-(s')\right\} = 0$$

where the curly brackets are the anti-commutator defined in the first of equations (4).

For the electron-positron field the above commutation rules are equivalent to

$$(7)\quad \left\{A^{\varepsilon}(s),\, A^{\varepsilon'}(s')\right\} = \left\{A^{-\varepsilon}(s),\, A^{-\varepsilon'}(s),\right\} = 0$$

$$(8)\quad \left\{A^{\varepsilon}(s),\, A^{-\varepsilon'}(s')\right\} = \delta(s',s) \quad .$$

In (7) and (8) the operators $A^{\pm\varepsilon}(s)$ are to be regarded as functions of s in terms of the superscript as well as in terms of the parenthesis, i.e. $A^{\pm\varepsilon}(s) = A^{\pm}(s)$ for values of s corresponding to positive energy; $A^{\pm\varepsilon}(s) = A^{\mp}(s)$ for values of s corresponding to negative energies. The commutation rules (7) and (8) are sometimes more convenient than (4), (5), and (6).

Biquantized Operators

Having introduced annihilation and creation operators we can now construct various operators which operate on the states of the field from operators of single particle quantum theory. In the case of a simple fermion field (see e.g. [1]) a well-defined biquantized operator $[L]_\Lambda$ can be constructed from a single particle operator L in the following way

$$\begin{aligned} [L]_\Lambda &= \int \Lambda^+(s) L^S \Lambda^-(s)\, ds \\ &= \iint L^S(s,s') \Lambda^+(s) \Lambda^-(s')\, ds\, ds' \end{aligned} \tag{9}$$

where L^S is the operator given in the spectral representation of the operator S and where $L(s,s')$ is defined by

$$L^S f(s) = \int L^S(s,s') f(s')\, ds' \tag{10}$$

$f(s)$ being a single particle state in the S representation. It is essential that the creation operator and annihilation operators appear in the order shown in (9) for operators such as $[L]_\Lambda$ to be defined. This fact motivates the following definition for biquantized operators which was suggested by K. O. Friedrichs for the case of the electron-positron field

$$(11) \qquad [L]_A = \iint L^S(s,s')\varepsilon' \Big[A^{\varepsilon}(s)\, A^{-\varepsilon'}(s')\Big]_{\varepsilon'} ds\, ds'$$

where

$$(12) \qquad \begin{aligned} [A^{\varepsilon}(s)A^{-\varepsilon'}(s')]_{\varepsilon'} &= A^{\varepsilon}(s)A^{-\varepsilon'}(s') \quad \text{for} \quad \varepsilon' > 0 \\ &= A^{-\varepsilon'}(s')A^{\varepsilon}(s) \quad \text{for} \quad \varepsilon' < 0. \end{aligned}$$

This prescription for obtaining biquantized operators from single particle operators makes it unnecessary to discuss the "sea" of positrons. Furthermore, biquantized operators obtained in this way have the desirable property that the expectation value of any such biquantized operator Γ is $\left(\Psi_o, \Gamma \Psi_o\right) = 0$ where Ψ_o is the vacuum state.

Let us now consider some examples of the use of (11). Let L be the Dirac Hamiltonian $\Omega^{(o)}$. Then in this case

$$(13) \qquad L^S(s,s') = \omega(s)\, \delta(s,s')$$

and hence

$$(14) \qquad [\Omega^{(o)}]_A = \int |\omega(s)| A^+(s)A^-(s)ds \quad .$$

It can be shown that if the state Ψ is represented by $\Psi \overset{}{\underset{N}{<\!-\!-\!-\!>}} \left\{\psi_n(s)_n\right\}$ then

(15) $$[\Omega^{(0)}]_A \Psi \underset{N}{\longleftrightarrow} \left\{ \sum_{j=1}^{n} |\omega(s'_j)| \psi_n(s)_n \right\} .$$

Note that only the absolute value of $\omega(s)$ is involved in (14) and (15), so that this operator has a positive spectrum. $[\Omega^{(0)}]_A$ is taken to be the Hamiltonian of the electron-positron field when no electromagnetic field is present. We shall denote $[\Omega^{(0)}]_A$ by $H^{(0)}$.

Let us now consider L to be the operator EI where I is the unit operator. Then

$$L^S(s,s') = \varepsilon\delta(s,s')$$

and

(16) $$[EI]_A = \int A^+(s)A^-(s)ds = N$$

where N is the number operator. Of course, as in the general case for fermions we have

(17) $$N\Psi \underset{N}{\longleftrightarrow} \left\{ n\psi_n(s)_n \right\}$$

which follows from (1) and (2).

To define the number operators for the number of electrons and positrons we must define two projection operators $\delta_{E,+1}$ and $\delta_{E,-1}$ which will play an essential role in much of what follows. The operator $\delta_{E,+1}$ is defined

in the single particle theory in terms of the S representation discussed above as follows. Given a single particle state Ψ whose representer in the S representation is f(s),

$$\Psi \underset{S}{<\!\!--\!\!-\!\!>} f(s) \quad ,$$

then $\delta_{E,+1}\Psi$ has the representer

$$\delta_{E,+1}\Psi \underset{S}{<\!\!--\!\!-\!\!>} \delta_{\varepsilon,+1} \; f(s)$$

where $\delta_{\varepsilon,+1}$ is the ordinary Kronecker I-function which is 1 when $\varepsilon = +1$ and zero if $\varepsilon = -1$. Thus

$$\delta_{\varepsilon,+1} \, f(s) = f(s)$$

for values of s corresponding to positive energy

$$\delta_{\varepsilon,+1} \, f(s) = 0$$

for values of s corresponding to negative energy. The operator $\delta_{E,-1}$ is defined analogously.

Now, if Ψ is a single particle state, $\left(\Psi, \delta_{E,+1}\Psi\right)$ is the probability that the electron has a positive energy. Similarly $\left(\Psi, \delta_{E,-1}\Psi\right)$ is the probability of the particle having a negative energy. Hence we take N_+ , the operator for the number of electrons to be

(18) $$N_+ = [\delta_{E,+1} EI]_A = \int A^+(s)A^-(s)\delta_{\varepsilon,+}\, ds$$

and N_- , the operator for the number of positrons to be

(19) $$N_- = [\delta_{E,-1} EI]_A = \int A^+(s)A^-(s)\delta_{\varepsilon,-}\, ds$$

It is seen that

(20) $$N_+ + N_- = N$$

Perhaps it would be useful to exhibit an eigenstate of N_+ and N_- in order that these operators may be more clearly understood. Consider a function $f_{+-}(s_1,s_2)$ which vanishes when s_1 has a value corresponding to a negative energy and which also vanishes when s_2 has a value corresponding to a positive energy. Consider next the function

(21) $$\psi_2(s_1,s_2) = f_{+-}(s_1,s_2) - f_{+-}(s_2,s_1)$$

The state

(22) $$\Psi_2 \underset{N}{\longleftrightarrow} \left\{0,\ 0,\ \psi_2(s_1,s_2),\ 0,\ 0\ \ldots\right\}$$

is an eigenstate of N, N_+ and N_- simultaneously. For from (17) it is clear that

(23) $$N\Psi_2 = 2\Psi$$

so that Ψ_2 is an eigenstate of N with eigenvalue 2. Also from (18), (1) and (2) it is seen that

(24) $$N_+\Psi_2 = \Psi_2$$
$$N_-\Psi_2 = \Psi_2$$

Hence Ψ_2 is an eigenstate of N_+ and N_- with an eigenvalue 1. One interprets Ψ_2 as the state of a field consisting of two particles, one of which is an electron, the other of which is a positron. In a similar way one can construct other eigenstates of N_+ and N_- .

We shall now construct the biquantized charge and current operators and the corresponding charge and current density operators.

The single-particle charge operator is e where e is the electron charge. The function $L^S(s,s')$ is thus

(25) $$L^S(s,s') = e\delta(s,s')$$

and the biquantized charge operator, which we denote by Q is

(26) $$Q = [e]_\Lambda = e(N_+ - N_-) .$$

This definition is a natural one in view of the fact that N_+ and N_- are the number operators for electrons and positrons respectively. It might be noted that the state Ψ_2 defined above is an eigenstate of Q with eigenvalue zero, so that Ψ_2 corresponds to a field with zero total charge.

The single particle current operator is eV_i where V_i is i'th component of n the velocity operator. But from the Dirac theory of the electron

$$V_i = \alpha_i$$

so that the i'th component of the single particle current operator is $e\alpha_i$. We write the function $L^S(s,s')$ as $e\alpha_i^S(s,s')$ and the i'th component of the biquantized current operator which we denote by V_i is

$$J_i = e\iint \epsilon'\alpha_i^S(s,s')\,[A^{+\epsilon}(s)A^{-\epsilon'}(s')]_{\epsilon'}\,ds\,ds' \tag{27}$$

To obtain the charge and current density operators, one is compelled to use the x-representation. We shall discuss this representation shortly. The single-particle charge density operator which gives the density at x' is $e\delta(X - x')$, X being the collection of operators X, Y, Z. F or the single particle case $\delta(X - x')$ is defined in terms

of the x-representation. If the single-particle state Ψ has as its x representer the function $f(x)$ $(x = \{x, y, z\})$ then $\delta(X - x')\Psi$ has as its representer $\delta(x - x')f(x)$ in which $\delta(x - x')$ is the three-dimensional delta function. The one-particle charge density is clearly highly singular. Our choice of the charge density operator is motivated by the fact that the one-particle matter density probability at x' is just the expectation value of $\delta(X - x')$.

The biquantized current operator is then

$$\rho(x') = [e\delta(X - x')]_A \tag{28}$$

The biquantized current density operator is similarly

$$j_i(x') = [e\alpha_j\delta(X - x')]_A \tag{29}$$

Equations (28) and (29) will be discussed more fully shortly. It will be later shown that

$$\int \rho(x')\, dx' = Q$$

$$\int j_i(x')\, dx' = J_i$$

if the integrations are carried out over all space, and the definitions (28) and (29) will be compared with the usual definitions. It should be noted that the vacuum expectation values

of $\rho(x)$ and $j_i(x)$ as defined above are zero as explained previously, whereas the vacuum expectations of the more conventional current and charge densities yield ambiguously defined constants.*

Many other biquantized operators can be defined in a similar fashion. However, we shall restrict our discussion to those introduced above.

---

* The author is indebted to Dr. B. Zumino for this simple method of obtaining the charge and current density operators. The author previously obtained the same result for these densities by a somewhat more cumbersome procedure.

The X - Representation

Thus far we have restricted ourselves to a number representation using a set of quantization variables in which the Hamiltonian of a field-free single electron is spectrally (or diagonally) represented. The reason for using this set of quantization variables is that the sign of the energy, which plays such an important part in the commutation rules for the annihilation and creation operators and in the definition of biquantized operators, is either one of the variables s or can be made a function of the variables, depending on which set of quantization variables is chosen (see page 3). The use of other sets of quantization variables for setting up a number representation would be very clumsy.

However, there are two operators which are often used with a set of quantization variables, corresponding to the x-representation of the annihilation and creation operators in the conventional number representation. This operator is important because in the Heisenberg picture, the differential equation for this operator, which is of the same form as the Dirac equation for a single electron in the Schrödinger picture, implicitly gives the Hamiltonian for the electron-positron field interacting with an external electromagnetic field. To study this operator, we shall

review briefly the x-representation of states for a single electron.

In the x-representation for the one electron problem, states are represented by "spinors" which are functions of x and another variable μ. The variable μ is restricted to four values which may be taken as 1, 2, 3, 4. States in this representation can be written as $f(x,\mu)$. The inner product of two states in this representation has the form

$$\sum_{\mu} \int \overline{f^{(1)}}(x,\mu)f(x,\mu)dx \quad .$$

The complete set of commuting variables which are spectrally represented in this way could be given, but for our purposes it is not necessary to discuss this matter in detail.

The quantization variables s discussed earlier correspond to the use of the energy representation. If a state Ψ is represented by $f(x,\mu)$ in the x-representation and by $g(s)$ in the energy representation $f(x,\mu)$ and $g(s)$ are related by a linear transformation which can be expressed as an integral operator with the kernel $\chi^{(o)}(x,\mu;s)$ or

$$f(x,\mu) = \int \chi^{(o)}(x,\mu;s)g(s)ds \tag{30}$$

The functions $\chi^{(o)}(x,\mu;s)$ are nothing other than the

eigenfunctions of $\Omega^{(0)}$ which are given in the textbooks, i.e.

$$(31)\qquad \Omega^{(0)x,\mu}\ \chi^{(0)}(x,\mu;s) = \omega(s)\ \chi^{(0)}(x,\mu;s)$$

where $\Omega^{(0)x,\mu}$ is the Dirac Hamiltonian in the x-representation

$$(32)\qquad \Omega^{(0)x,\mu} = i \sum_i \alpha_i^\mu \frac{\partial}{\partial x_i} - \beta^\mu m$$

and satisfy the orthogonality conditions

$$(33)\qquad \sum_\mu \int \overline{\chi^{(0)}}(x,\mu;s)\ \chi^{(0)}(x,\mu;s')dx = \delta(s,s')$$

$$(34)\qquad \int \overline{\chi^{(0)}}(x,\mu;s)\ \chi^{(0)}(x',\mu';s)ds = \delta(x,x')\delta_{\mu\mu'} .$$

α_i^μ , β^μ are the usual Dirac matrices.

Let us now return to the electron-positron field theory. The operators which we wish to introduce in the x-representation are $\Xi^-(x,\mu)$ and $\Xi^+(x,\mu)$ which are related to the annihilation and creation operators in a manner analogous to (30). We define these operators by

$$(35)\qquad \begin{aligned} \Xi^-(x,\mu) &= \int \chi^{(0)}(x,\mu;s)\ \Lambda^{-\varepsilon}(s)ds \\ \Xi^+(x,\mu) &= \int \overline{\chi}^{(0)}(x,\mu;s)\ \Lambda^{+\varepsilon}(s)ds \end{aligned}$$

The orthogonality condition (33) leads to the relation

$$\Lambda^{-\varepsilon}(s) = \sum_{\mu} \int \bar{\chi}^{(o)}(x,\mu;s)\, \Xi^{-}(x,\mu)\, dx \tag{36}$$

$$\Lambda^{+\varepsilon}(s) = \sum_{\mu} \int \chi^{(o)}(x,\mu;s)\, \Xi^{+}(x,\mu)\, dx$$

as in an earlier context the operators $\Lambda^{\pm\varepsilon}(s)$, are to be considered as functions of the superscript ε as well as of the argument s.

The operators Ξ^{-}, Ξ^{+} obey the following commutation rules, as can be shown from (7) and (8)

$$\left\{\Xi^{+}(x,\mu),\ \Xi^{-}(x',\mu')\right\} = \delta(x - x')\ \delta_{\mu\mu'} \tag{37}$$

$$\left\{\Xi^{+}(x,\mu),\ \Xi^{+}(x',\mu')\right\} = \left\{\Xi^{-}(x,\mu),\ \Xi^{-}(x',\mu')\right\} = 0$$

We are now in a position to express our biquantized operators in terms of Ξ^{-}, Ξ^{+} and compare these operators with the more conventional operators obtained from the Lagrangian formalism.

The energy operator $H^{(o)}$ can be expressed in terms of Ξ^{+}, Ξ^{-}, as follows by means of (14) and (36).

$$H^{(o)} = \sum_{\mu} \Xi^{+}(x,\mu)\ \Omega^{(o)x,\mu}\ \Xi^{-}(x,\mu)\, dx \quad . \tag{38}$$

$$\cdot \quad - \delta(s - s) \int ds'\omega(s')\ I_{\varepsilon',-1}$$

This expression differs from the usual expression which is the first term of the above expression by the highly singular constant $\delta(s-s)\int ds'\omega(s')\ \delta_{\varepsilon',-1}$ which can be interpreted as the energy of the "sea" which has to be subtracted to make $H^{(o)}$ a positive definite operator.

The current density operator also has an interesting form in terms of the operators Ξ^-, Ξ^+.

First of all, the function $L^S(s,s')$ corresponding to the operator $\delta(X - x')$ is

$$\sum_{\mu}\int\overline{\chi}^{(o)}(x,\mu;s)\delta(x - x')\ \chi^{(o)}(x,\mu;s')\ dx$$

$$= \sum_{\mu}\overline{\chi}^{(o)}(x',\mu;s)\ \chi^{(o)}(x',\mu;s').$$

Thus

$$\rho(x') = e\sum_{\mu}\iint\overline{\chi}^{(o)}(x',\mu;s)\,\chi^{(o)}(x',\mu;s')\ \cdot \tag{28a}$$

$$\cdot\ \ \varepsilon'\ [\Lambda^{+\varepsilon}(s)\Lambda^{-\varepsilon'}(s')]_{\varepsilon'}\ dsds'\ .$$

From this form of the expression for $\rho(x')$ we can show that $\int\rho(x')dx' = Q$. If we take the integral $\int\rho(x')dx'$ we see that by interchanging the order of integration

$$\int\rho(x')dx' = e\iint\left(\sum_{\mu}\int\overline{\chi}^{(o)}(x',u;s)\,\chi^{(o)}(x',\mu;s')dx'\right)\ \cdot$$

$$\cdot\ \ \varepsilon'\ [\Lambda^{+\varepsilon}(s)\Lambda^{-\varepsilon'}(s')]_{\varepsilon'}\ dsds'$$

$$= e\iint\delta(s,s')\varepsilon'[\Lambda^{+}(s)\Lambda^{-\varepsilon'}(s')]_{\varepsilon'}\ dsds' = [e]_{\Lambda}$$

$$= Q$$

Hence the charge density operator has the property that its integral over all space is the charge operator as desired.

Now to obtain $\rho(x)$ in terms of Ξ^-, Ξ^+ we substitute the expressions (36) for $\Lambda^{\pm\varepsilon}$ in (28a) and find after some rearrangement that $\rho(x')$ can be expressed in terms of any of the equivalent forms below

(39)

$$\rho(x) = e \sum_{\mu} \Xi^+(x,\mu)\Xi^-(x,\mu) - e \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\chi^{(0)}(\mu;s)\delta_{\varepsilon,-1}ds$$

$$= -e \sum_{\mu} \Xi^-(x,\mu)\Xi^+(x,\mu) + e \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\chi^{(0)}(\mu;s)\delta_{\varepsilon,+1}ds$$

$$= \frac{1}{2} e \sum_{\mu} \left[\Xi^+(x,\mu)\Xi^-(x,\mu) - \Xi^-(x,\mu)\, \Xi^+(x,\mu)\right]$$

$$+ \frac{1}{2} e \left[\sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\, \chi^{(0)}(\mu;s)\delta_{\varepsilon,+1}\, ds - \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\, \chi^{(0)}(\mu;s)\delta_{\varepsilon,-1}ds \right]$$

where the functions $\chi^{(0)}(\mu;s)$ are defined by

$$\chi^{(0)}(x,\mu;s) = e^{ip.x}\,\chi^{(0)}(\mu;s) \qquad (40)$$

p.x being the inner product of the momentum and position vector. These functions are called the "spinor coefficients" of the plane wave $e^{ip.x}$ and are the usual ones given in

text books.

It should be noted that

$$\int \bar{\chi}^{(0)}(\mu;s)\, \chi^{(0)}(\mu;s)ds = \int \bar{\chi}^{(0)}(x,\mu;s)\, \chi^{(0)}(x,\mu;s)ds$$

$$= \delta_{\mu,\mu}\, \delta(x - x) = \delta\ (0)$$

from the orthogonality relations between the eigenfunctions. Thus the terms independent of x in the definition of current are highly singular functions, of the nature of $\delta(0)$.

By expressing $\Xi^{-}(x,\mu)$, $\Xi^{+}(x,\mu)$ in terms of $\Lambda^{\pm}(s)$ we find that for states represented by $\underline{\Psi} \underset{\overline{N}}{\longleftrightarrow} \left\{\psi_n(s)_n\right\}$ we have the expectation value of $\rho(x)$ given by

$$(41)\quad \left\{\underline{\Phi},\ \rho(x)\ \underline{\Phi}\right\}$$

$$= \sum_{\mu} \sum_{n=0}^{\infty} \Bigg\{ (n+1) \int (ds)_n \Bigg[\Big| \int \chi^{(0)}(x,\mu;s')\psi_{n+1}(s)_n(s')\delta_{\varepsilon',+1}ds' \Big|^2$$

$$+ \Big| \int \overline{\chi^{(0)}}(x,\mu;s')\psi_{n+1}(s)_n(s')\ \delta_{\varepsilon',-1}ds' \Big|^2 \Bigg]$$

$$+ 2R \sqrt{n+1}\ \sqrt{n+2} \iiint (ds)_n ds'\ ds''\ \overline{\Psi}_n(s)_n\psi_{n+2}(s)_n(s')(s'')\ \cdot$$

$$\cdot\ \overline{\chi^{(0)}}(x,\mu;s')\ \chi^{(0)}(x,\mu;s'')\ \delta_{\varepsilon',-1}\delta_{\varepsilon'',+1}$$

where R means the real part. This expression is finite for a large class of states $\underline{\Psi}$. Hence $\rho(x)$ is a well

defined operator in a suitable subspace.

By virtue of our definition of biquantized operators, the operator $\rho(x)$ has the gratifying property that for states Ψ_0 which correspond to the vacuum state we have

$$(41a) \qquad \left(\Psi_0, \ \rho(x)\, \Psi_0\right) = 0.$$

That is, the vacuum expectation of the charge is zero. The usual form of the charge density operator due to Heisenberg is $\frac{1}{2}\, e \sum_{\mu} \left[\Xi^{+}(x,\mu)\, \Xi^{-}(x,\mu) - \Xi^{-}(x,\mu)\, \Xi^{+}(x,\mu) \right]$ which differs from our definition by the highly singular constant

$$\frac{1}{2}\, e \left[\sum_{\mu} \int \overline{\chi}^{(o)}(\mu;s)\, \chi^{(o)}(\mu;s)\, \delta_{\varepsilon,+1} ds - \sum_{\mu} \int \overline{\chi}^{(o)}(\mu;s)\, \chi^{(o)}(\mu;s)\, \delta_{\varepsilon,-1} ds \right]$$

so that the vacuum expectation of the usual charge density is just this constant. Various excuses are then required to permit one to set it equal to zero. But using the charge density operator of the present paper we obtain (41a) merely by expressing Ξ^{+} and Ξ^{-} in terms of Λ^{+} and Λ^{-} and taking the expectation value in a straightforward fashion.

By working in the x-representation we can understand more clearly the form of the current and current density operators. In equation (27) we can express $\alpha_i^S(s,s')$ by

$$\alpha_i^S(s,s') = \sum_{\mu} \int dx \bar{\chi}^{(o)}(x,\mu;s)\ \alpha_i^{\mu}\ \chi^{(o)}(x,\mu;s')$$

so that one may now express J_i explicitly in terms of the operators $\Lambda^{\pm}(s)$.

In the case of the current density operator, the function $L^S(s,s')$ in (29) is

$$e \sum_{\mu} \int dx\ \bar{\chi}^{(o)}(x,\mu;s)\ \delta(x - x')\ \alpha^{\mu}\ \chi^{(o)}(x,\mu;s')$$

$$= e \sum_{\mu} \bar{\chi}^{(o)}(x',\mu;s)\ \alpha^{\mu}\ \chi^{(o)}(x',\mu;s').$$

It is clear that

$$J_i = \int j_i(x')\ dx'$$

where the integration is taken over all space. In terms of the operators Ξ^- and $\Xi^{\dagger}$ we can express the current density as

$$(42) \qquad j_i(x)$$

$$\begin{aligned}
&= e \sum_{\mu} \Xi^{+}(x,\mu)\alpha_i^{\mu}\Xi^{-}(x,\mu) - e \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\alpha_i^{\mu}\chi^{(0)}(\mu;s)\delta_{\varepsilon,-1}ds \\
&= -e \sum_{\mu} \Xi^{-}(x,\mu)\bar{\alpha}_i^{\mu}\Xi^{+}(x,\mu) + e \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\alpha_i^{\mu}\chi^{(0)}(\mu;s)\delta_{\varepsilon,+1}ds \\
&= \frac{1}{2} e \sum_{\mu}\Big[\Xi^{+}(x,\mu)\alpha_i^{\mu}\Xi^{-}(x,\mu) - \Xi^{-}(x,\mu)\bar{\alpha}_i^{\mu}\,\Xi^{+}(x,\mu)\Big] \\
&\quad + \frac{1}{2} e \Big[\sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\ \alpha_i^{\mu}\chi^{(0)}(\mu;s)\ \delta_{\varepsilon,+1}ds \\
&\qquad\qquad - \sum_{\mu}\int \bar{\chi}^{(0)}(\mu;s)\ \alpha_i^{\mu}\chi^{(0)}(\mu;s)\ \delta_{\varepsilon,-1}ds\Big]
\end{aligned}$$

which differs from the Heisenberg current operator by a highly singular constant. As in the case of the charge density operator, the vacuum expectation of $j_i(x)$ is zero.

We shall show later that the current and charge density operators satisfy the equation of continuity in the Heisenberg picture.

Variation of States and Operators with Time.

Having indicated the spectral representation of $H^{(o)}$ the biquantized Hamiltonian for the field-free case we can solve the Schrödinger equation for the electron-positron field. The Schrödinger equation is

$$(43) \qquad i\,\frac{\partial \Psi(t)}{\partial t} = H^{(o)}\Psi(t)$$

for which the solution is

$$(44) \qquad \Psi(t) = e^{-itH^{(o)}}\Psi(0)$$

where $\Psi(0)$ is the state at time $t = 0$. If we use the number representation and write

$$(45) \qquad \Psi(0) \xleftrightarrow[N]{} \left\{\psi_n(s)_n\right\}$$

$$\Psi(t) \xleftrightarrow[N]{} \left\{\psi_n(s)_n(t)\right\}$$

we have from (44) and (15)

$$(46) \qquad \psi_n(s)_n(t) = e^{-it\left[\sum_{i=1}^{n}|\omega(s_i)|\right]}\psi_n(s)_n$$

Equation (46) represents the solution of (43) in a form useful for calculating expectation values. It is clear that the expectation value of N and $H^{(o)}$, the number operator and Hamiltonian respectively, are constant in time.

Let us now consider the Heisenberg picture. Any operator L given at time t = 0 is given at time t by

$$(47) \qquad L(t) = e^{-itH^{(o)}} L \, e^{itH^{(o)}}$$

For the annihilation and creation operators this leads to

$$(48) \qquad \begin{cases} A^{-}(s)(t) = e^{-it|\omega(s)|} A^{-}(s) \\ A^{+}(s)(t) = e^{+it|\omega(s)|} A^{+}(s) \end{cases}$$

or

$$(48a) \qquad A^{\mp\varepsilon}(s)(t) = e^{\mp it\,\omega(s)} A^{\mp\varepsilon}(s)$$

$A^{+}(s)$ and $A^{-}(s)$ being the creation and annihilation operators at time t = 0. Using (48) it can be seen that the number operators N, N_{+}, N_{-} are constant in time as is to be expected.

It is also seen that the charge operator Q(t) is constant in time. It can be shown that the current operator is also constant in time.

The operators $\Xi^-(x,\mu)(t)$ and $\Xi^+(x,\mu)(t)$ are taken to have the same relation to $\Lambda^{\pm}(s)(t)$ as $\Xi^-(x,\mu)$ and $\Xi^+(x,\mu)$ have to $\Lambda^{\pm}(s)$, i.e.

$$\begin{aligned} \Xi^-(x,\mu)(t) &= \int \chi^{(o)}(x,\mu;s)\, \Lambda^{-\varepsilon}(s)(t)\, ds \\ &= \int \chi^{(o)}(x,\mu;s)\, e^{-it\,\omega(s)}\, \Lambda^{-\varepsilon}(s)\, ds \end{aligned} \tag{35a}$$

$$\begin{aligned} \Xi^+(x,\mu)(t) &= \int \bar{\chi}^{(o)}(x,\mu;s)\, \Lambda^{+\varepsilon}(s)(t)\, ds \\ &= \int \bar{\chi}^{(o)}(x,\mu;s)\, e^{it\,\omega(s)}\, \Lambda^{+\varepsilon}(s)\, ds \end{aligned}$$

$$\begin{aligned} \Lambda^{-\varepsilon}(s)(t) &= \sum_{\mu} \int \bar{\chi}^{(o)}(x,\mu;s)\, \Xi^-(x,\mu)(t) \\ \Lambda^{+\varepsilon}(s)(t) &= \sum_{\mu} \int \chi^{(o)}(x,\mu;s)\, \Xi^+(x,\mu)(t)\, dx \end{aligned} \tag{36a}$$

It is clear from (35a) and (31) that Ξ^-, Ξ^+ obey the Dirac equations for the field free case

$$\begin{aligned} \Omega^{(o)x,\mu}\, \Xi^-(x,\mu)(t) &= i \frac{\partial}{\partial t} \Xi^-(x,\mu)(t) \\ \bar{\Omega}^{(o)x,\mu}\, \Xi^-(x,\mu)(t) &= -i \frac{\partial}{\partial t} \Xi^+(x,\mu)(t) \end{aligned} \tag{49}$$

It can now be shown that the current and charge density operators defined above do indeed formally satisfy the equation of continuity as required

$$\frac{\partial \rho(x)(t)}{\partial t} + \mathrm{div}\, j(x) = 0$$

the proof being essentially the same as in the one-particle theory if one puts "boundary conditions" on the operators Ξ^-, Ξ^+.

Interaction of the Electron-Positron Field with an Electromagnetic Field

The way the interaction with the electromagnetic field appears is through an equation for $\Xi^-(x,\mu)(t)$ in the Heisenberg picture. The equation for $\Xi^-(x,\mu)(t)$ is

$$(50) \qquad \Omega^{x,\mu}\, \Xi^-(x,\mu)(t) = i \frac{\partial}{\partial t} \Xi^-(x,\mu)(t)$$

$$\bar{\Omega}^{x,\mu}\, \Xi^+(x,\mu)(t) = -i \frac{\partial}{\partial t} \Xi^+(x,\mu)(t)$$

where

$$(51) \qquad \Omega^{x,\mu} = \Omega^{x,\mu(0)} + e\, \Omega^{(1)}$$

$$(52) \qquad \Omega^{x,\mu(1)} = \sum_i \alpha_i^\mu A_i(x) + \phi(x)$$

where in our problem $A_i(x)$ and $\phi(x)$ are the components of the vector potential and scalar potential and are given functions of x. The form of $\Omega^{(1)}$ given in (50) corresponds to an interaction with a point electron. Sometimes it will be convenient to think of $\Omega^{(1)}$ having a more general form, so that the interaction would be that corresponding to an electron of finite size, which in the limit would take the form (50).

The commutation rules for $\Xi^-(x,\mu)(o) = \Xi^-(x,\mu)$, $\Xi^+(x,\mu)(o) = \Xi^+(x,\mu)$, $A^+(s)$, $A^-(s)$ are taken to be those given previously. The relations (35a), (36a) which relate $\Xi^-(x,\mu)(t)$, $\Xi^+(x,\mu)(t)$ to $A^{\pm}(s)(t)$ are taken as being valid

even with the interaction.

However, the equations (46) and (48) which give $\Lambda^{\pm}(s)(t)$ in the Heisenberg picture and $\psi_n(s)_n(t)$ in the Schrödinger picture as functions of time in terms of these quantities when $t = 0$ are no longer valid. In fact the problem which we wish to solve is how to express these quantities as functions of time in terms of the quantities when $t = 0$. Then, if, for example, one uses the Schrödinger picture, it is possible to find expectation values of various operators $\Gamma\left(\Psi(t),\Gamma\Psi(t)\right)$ as functions of time when these expectation values are known at time $t = 0$.

There are several equivalent ways of finding $\Lambda^{\pm}(s)(t)$ or $\psi_n(s)_n(t)$ in terms of $\Lambda^{\pm}(s)$ or $\psi_n(s)_n$. Among them two of the most obvious are the following:

1) One may solve the operator equation (50) as though $\Xi^{-}(x,\mu)(t)$ were an ordinary function, through the use of an influence function. Then one could find $\Lambda^{\pm}(s)(t)$ in terms of $\Lambda^{\pm}(s)$ using (35a). Since one knows how $\Lambda^{\pm}(s)$ operate on the Heisenberg states, one would be able to see how $\Lambda^{\pm}(s)(t)$ operate on these states and thus it would be possible to find expectation values of biquantized operators constructed with the aid of $\Lambda^{\pm}(s)(t)$. This point of view is close to that exemplified by the work of Yang and Feldman [2]. The disadvantage of this method is that the influence

function involves singularities which must be handled very carefully. To obviate the necessity of dealing with such influence functions directly, we use a second method, which, however, is valid only for the condition that $A_i(x)$, $\phi(x)$ are time independent.

2) One considers the eigenfunctions $\chi(x,\mu;v)$ of the Hamiltonian Ω. Here v consists of a complete set of commuting variables such that the energy ν is a function of them (or is one of them), i.e.

$$(53) \qquad \Omega^{x,\mu}\,\chi(x,\mu;v) = \nu(v)\,\chi(x,\mu;v) \quad .$$

One can then define annihilation and creation operators $B^{\pm}(v)(t)$ in a manner analogous to equations (35a), (36a). Furthermore an energy operator, constant in time, can be defined for the electron-positron field through the biquantization of Ω using the operator $B^{\pm}(v)(t)$. The new number operator $M = \int B^{+}(v)(t)B^{-}(v)(t)dv$ will be constant in time. One has thereby introduced "eigen-particles" or "proper particles". In this representation the Schrödinger equation can be solved. It is then only necessary to find a transformation from the N-representation to the M-representation, which is equivalent to finding a canonical transformation, after which it is possible to obtain the representers $\psi_n(s)_n(t)$ in terms of $\psi_n(s)_n$. The lectures of Kroll [3] exemplifies this point of view. In the next section we shall introduce the proper-particle representation.

The Proper Particle Representation

We shall assume that the spectrum of Ω is the same as that of $\Omega^{(o)}$. Then v is a set of variables whose spectrum has a range of values the same as that of s. Moreover, the function $\nu(v)$ is the same as $\omega(v)$. $\nu(v) \equiv \omega(v)$. It might appear that this assumption is very restrictive, but this assumption is implicit in much of the work done in modern field theory. What follows can be generalized considerably so that any spectrum may be assumed for Ω, the special assumption above being used principally to make the problem definite.

We introduce annihilation and creation operators $B^{\pm}(v)$ and a number operator $M = \int B^{+}(v)B^{-}(v)dv$. These operators are defined with respect to a representation

$$\Psi \underset{M}{\longleftrightarrow} \left\{ \zeta_n(v)_n \right\}$$

in a manner similar to the definition of the operators $A^{\pm}(s)$ on Ψ with respect to the representation $\Psi \underset{N}{\longleftrightarrow} \left\{ \psi_n(s)_n \right\}$ (see equations (1) and (2)). The operators $B^{\pm}(v)$ obey commutation relations similar to (7) and (8). We shall denote the sign of the energy Ω by ι (iota). The commutation rules are then

$$(54) \quad \left\{B^{\iota'}(v'), B^{\iota}(v)\right\} = \left\{B^{-\iota'}(v'), B^{-\iota}(v)\right\} = 0$$

$$\left\{B^{\iota'}(v'), B^{-\iota}(v)\right\} = \delta(v,v') \quad .$$

As with the operators $A^{\pm}(s)$, we can biquantize with respect to $B^{\pm}(v)$ an operator L defined for first quantization. Denoting such a biquantized operator by $[L]_B$ we write

$$(55) \quad [L]_B = \iint L^V(v,v')\iota \;\; [B\ (v)B^{-\iota'}(v')]_{\iota'}\ dv\ dv'$$

where $L^V(v,v')$ is the integral operator defined by

$$(56) \qquad L^V f(v) = \int L^V(v,v')\ f(v')dv'$$

$f(v)$ being a single particle state in the Ω representation. Also

$$(57) \quad [B^{\iota}(v)B^{-\iota'}(v')]_{\iota'} = B^{\iota}(v)\ B^{-\iota'} \qquad \text{for } \iota' > 0$$

$$= B^{-\iota'}(v)\ B^{\iota}(v) \quad \text{for } \iota' < 0.$$

In particular the energy of the field is given as

$$(58) \qquad [\Omega]_B = \int |\omega(v)|\ B^{+}(v)\ B^{-}(v)\ dv.$$

We denote the operator $[\Omega]_B = H$.

It can be shown that if $\Psi \underset{M}{\longleftrightarrow} \left\{\zeta_n(v)_n\right\}$ then $H\Psi$ is represented by

$$H\Psi \underset{M}{\longleftrightarrow} \left\{ \sum_{i}^{n} |\omega(v_i)| \, \zeta_n(v)_n \right\} .$$

Thus H has the same spectrum as $H^{(o)}$. In fact, this is one of the objectives in setting up the Hamiltonian as we have.

Though this assumption for the Hamiltonian seems special, it should be recollected that most recent writers have implicitly made the same assumption. For example, Schwinger [4] seeks a canonical transformation which transforms the Hamiltonian of an electron-positron field interacting with an electromagnetic field to the Hamiltonian of an electron-positron field without the electromagnetic field. The assumption that such a canonical transformation exists implies that the two Hamiltonians have the same spectrum. It might be noted also that when, as is often done, the electromagnetic potential is taken to be that for the Coulomb potential, this condition on the spectrum is not satisfied, and a more detailed discussion is necessary.

If we write the solution of the Schrödinger equation as

$$(59) \qquad \Psi(t) = e^{-it\,H}\,\Psi$$

and represent $\Psi(t)$ as $\Psi(t) \underset{M}{\longleftrightarrow} \left\{ \zeta_n(v)_n(t) \right\}$

$$(60) \qquad \zeta_n(v)_n(t) = e^{-it\left\{\sum_{i=1}^{n} |\omega(v_i)|\right\}} \zeta_n(v)_n \quad .$$

In the Heisenberg picture one can show

$$(61) \qquad B^-(v)(t) = B^-(v)\, e^{-it|\omega(v)|}$$

$$B^+(v)(t) = B^+(v)\, e^{+it|\omega(v)|}$$

in analogy to (48). Hence the number operator $M = \int B^+(v)(t) B^-(v)(t) dv$ is a constant as is the Hamiltonian $H = \int |\omega(v)| B^+(v)(t)\, B^-(v)(t) dv$.

The operators $B^{\pm}(v)$ can be related to the operators $\Xi^-(x,\mu)$ and $\Xi^+(x,\mu)$ by means of the eigenfunctions $\chi(x,\mu;v)$ of the Hamiltonian Ω. The relations (given for any time t in the Heisenberg representation) are in analogy to (35a) and (36a)

$$(62) \qquad \Xi^-(x,\mu)(t) = \int \chi(x,\mu;v)\, B^{-\prime}(v)(t)\, dv$$

$$\Xi^+(x,\mu)(t) = \int \bar{\chi}(x,\mu;v)\, B^{+\prime}(v)(t)\, dv$$

$$(63) \qquad B^{-\prime}(v)(t) = \sum_{\mu} \int \bar{\chi}(x,\mu;v)\, \Xi^-(x,\mu)(t)\, dx$$

$$B^{+\prime}(v)(t) = \sum_{\mu} \int \chi(x,\mu;v)\, \Xi^+(x,\mu)(t)\, dx \quad .$$

It is clear that our definition of energy of the field (58) which leads to the relations for $B^{\pm}(v)(t)$ as given by (61) is correct in the sense that from (62) we have

$$\Omega^{x,\mu}\, \Xi^{-}(x,\mu)(t) = i \frac{\partial}{\partial t}\, \Xi^{-}(x,\mu)(t) \qquad \text{and}$$

$$\Omega^{x,u}\, \Xi^{+}(x,\mu)(t) = -i \frac{\partial}{\partial t}\, \Xi^{+}(x,\mu)(t)$$

as is necessary (see discussion of equation (50)).

Equations (62) and (63), and (35a) and (36a) provide the relation between the operators $B^{\pm}(v)(t)$ and $A^{\pm}(s)(t)$. The relations are

$$(64) \qquad B^{-\ell}(v)(t) = \int u(v;s)\, A^{-\varepsilon}(s)(t)\,ds$$

$$B^{+\ell}(v)(t) = \int \bar{u}(v;s)\, A^{+\varepsilon}(s)(t)\,ds$$

and

$$(65) \qquad A^{-\varepsilon}(s)(t) = \int \bar{u}(v;s)\, B^{-\ell}(v)(t)\,dv$$

$$A^{+\varepsilon}(s)(t) = \int u(v;s)\, B^{+\ell}(v)(t)\,dv$$

where

$$(66) \qquad u(v;s) = \sum_{\mu} \int \bar{\chi}(x,\mu;v)\, \chi^{(0)}(x,\mu;s)\, dx$$

and is the kernel of the integral transformation which transforms the single particle representers in the spectral representation of $\Omega^{(0)}$ in the representer in the spectral representation of Ω. This point will be discussed more fully shortly.

It should be noted that $u(v;s)$ satisfies the orthogonality conditions

$$\int \bar{u}(v;s)\, u(v;s')\, dv = \delta(s,s')$$

$$\int \bar{u}(v;s)\, u(v';s)\, ds = \delta(v,v') \quad .$$

Using relations (64) and (65) and the above orthogonality conditions, we can express the charge operator $Q(t)$ in terms of $B^{\pm}(v)(t)$. It is found that $Q(t)$ is constant in time. Hence the charge operator is a constant of the motion and electrons and positrons are created in pairs. In the original picture of Dirac, these statements were obvious. In the present description, the proof sketched above is required.

It is also easily seen that the current density and charge density operators satisfy the equation of continuity even when an external electromagnetic field is present.

It is perhaps interesting to express H as the sum of $H^{(o)}$ and a perturbation. For this purpose we shall express the perturbation in terms of the operators Ξ^-, Ξ^+ . Our result can be written as

$$H = H^{(o)} + e \sum_{\mu} \int \Xi^{+}(x,\mu)\, \Omega^{(1)x,\mu}\, \Xi^{-}(x,\mu)\, dx$$

Using the expression (52) for $\Omega^{(1)x,\mu}$ we see that the perturbation is

$$\sum_{\mu} \int \Xi^{+}(x,\mu)\ \Xi^{-}(x,\mu)\ \phi(x)\ dx$$

$$+ \sum_{i} \sum_{\mu} \int \Xi^{+}(x,\mu)\ \alpha_i^{\mu}\ \Xi^{-}(x,\mu)\ A_i(x)\ dx$$

$$= \int \rho(x)\ \phi(x)\ dx + \sum_{i} j_i(x)\ A_i(x)\ dx$$

$$+ \sum_{\mu} \int \bar{\chi}^{(0)}(\mu;s)\ \chi^{(0)}(\mu;s)\ \delta_{\varepsilon,-1}\ dx \int \phi(x)\ dx$$

$$- \sum_{i} \sum_{\mu} \int \bar{\chi}^{(0)}(\mu;s)\ \alpha_i^{\mu}\ \chi^{(0)}(\mu;s)\ \delta_{\varepsilon,-1}\ dx \int A_i(x)\ dx$$

which, except for the last two terms, is the usual form of the perturbation. The last two terms, though highly singular, will not affect expectation values of various biquantized operators inasmuch as these terms are merely constants.

It might be noted that if we had been dealing with an electron field the relation (64) would have been replaced by

$$\text{(64a)} \qquad B^{-}(v)(t) = \int u(v;s)\ A^{-}(s)(t)\ ds$$

$$B^{+}(v)(t) = \int \bar{u}(v;s)\ A^{+}(s)(t)\ ds \quad .$$

From (64) and (65) one sees that

$$\Lambda^{-\varepsilon}(s)(t) = \int \bar{u}(v;s)\ B^{-\iota}(v)\ e^{-i\iota t|\omega(v)|}$$

$$= \int \Lambda^{-\varepsilon'}(s')ds' \int \bar{u}(v;s)\ u(v;s')\ e^{-i\iota t|\omega(v)|}\ dv \quad .$$

Likewise

$$\Lambda^{+\varepsilon}(s)(t) = \int \Lambda^{+\varepsilon'}(s')ds' \int u(v;s)\ \bar{u}(v;s')\ e^{i\iota t|\omega(v)|}\ dv$$

so that we have expressed $\Lambda^{\pm}(s)(t)$ in terms of $\Lambda^{\pm}(s)$ and therefore could find various expectation values as functions of the time in the N-representation since we now know how $\Lambda^{\pm}(s)(t)$ operates on states given in the N-representation.

However, we prefer to work in the Schrödinger picture because we are led to some interesting transformations which are closely analogous to those used in first quantization. These transformations may be interpreted as the canonical transformation which transforms the Hamiltonian H into $H^{(o)}$, i.e. the Hamiltonian of the electron-positron field with interaction into the Hamiltonian of the electron-positron field without interaction. Toward this end we shall now discuss the function u(v;s) defined by (66).

The Spectral Representation of Ω

Let us consider now the single electron problem with the Hamiltonian Ω. Let the state Ψ have a representer

$$\Psi \underset{v}{\longleftrightarrow} f(v)$$

in the spectral representation of Ω and a representer

$$\Psi \underset{s}{\longleftrightarrow} g(s)$$

in the spectral representation of $\Omega^{(o)}$. The solution of the Schrödinger equation is, of course,

$$\Psi(t) = e^{-it\Omega}\,\Psi \tag{67}$$

where $\Psi = \Psi(o)$.

If we write

$$\Psi(t) \underset{v}{\longleftrightarrow} f(v)(t)$$

we have the solution given by

$$f(v)(t) = e^{-it\omega(v)} f(v) \quad . \tag{68}$$

To find how the representer $g(s)(t)$ changes with time where $\Psi(t) \underset{s}{\longleftrightarrow} g(s)(t)$ we write the transformation between the representers in the Ω and $\Omega^{(o)}$ representation as an integral transformation with the kernel $u(v;s)$

(69) $$f(v) = \int u(v;s)\ g(s)\ dv$$

with u(v;s) satisfying the orthogonality condition

(70) $$\begin{cases} \displaystyle\int \bar{u}(v;s)\ u(v;s')dv = \delta(s,s') \\ \displaystyle\int \bar{u}(v';s)\ u(s;s)ds = \delta(v,v') \end{cases} .$$

The conditions (70) guarantee that

(71) $$\int |f(v)|^2 = \int |g(s)|^2 ds$$

i.e., lengths and inner products are preserved. From (68), (69) and (70),

(72) $$g(s) = \int \bar{u}(v;s)\ f(v)dv$$

(73) $$\begin{aligned} g(s)(t) &= \int \bar{u}(v;s)\ f(v)(t)dv \\ &= \int \bar{u}(v;s)\ e^{-it\omega(v)} f(v)dv \end{aligned}$$

or from (69)

(74) $$g(s)(t) = \int g(s')ds' \int \bar{u}(v;s)u(v;s')e^{-it\omega(v)}dv.$$

Equation (74) represents the solution of the Schrödinger equation in the $\Omega^{(o)}$ representation in terms of $g(s)(o) \equiv g(s)$.

We shall now show how the function u(v;s) may be determined.

We denote the operator Ω in the v representation by Ω^v and in the s representation by Ω^s. It is clear that

$$\Omega^s = \Omega^{(0)s} + e\Omega^{(1)s} \quad . \tag{75}$$

Now from the definition of Ω^v we have

$$\Omega^v f(v) = \omega(v)\, f(v) \tag{76}$$

and from the definition of $\Omega^{(0)s}$

$$\Omega^{(0)s} g(s) = \omega(s)\, g(s) \quad . \tag{77}$$

From (69), (75), (76) and (77) we have

$$\Omega^v f(v) = \int u(v;s)\ \Omega^s g(s)$$

or

$$\begin{aligned} \omega(v)f(v) &= \int u(v;s)[\omega(s)g(s) + e\Omega^{(1)s}g(s)]ds \\ &= \int \left\{\left(\omega(s) + e\overline{\Omega}^{(1)s}\right)u(v;s)\right\} g(s)ds \end{aligned} \tag{78}$$

$\Omega^{(1)}$ being Hermitian. But also

$$\Omega^v f(v) = \int \Omega^v u(v;s)g(s)ds$$

or

$$\omega(v)f(v) = \omega(v) \int u(v;s)g(s)ds \quad . \tag{79}$$

Since the state represented by $g(s)$ and $f(v)$ is arbitrary, we have from (78) and (79)

$$(80) \qquad \omega(v)u(v;s) = \omega(s)u(v;s) + c\overline{\Omega^{(1)s}}u(v;s)$$

where $\overline{\Omega^{(1)s}}u(v;s)$ means that $\overline{\Omega^{(1)s}}$ operates with respect to the variable s in $u(v;s)$.

It will be convenient to consider $\ddot{\Omega}^{(1)s}$ an integral operator with the kernel $\Omega^{(1)s}(s;s')$. Thus

$$(81) \qquad \Omega^{(1)s}g(s) = \int \Omega^{(1)s}(s;s')g(s')ds' \quad .$$

The kernel $\Omega^{(1)\ s}(s;s')$ is nothing other than the matrix element of $\Omega^{(1)}$ in the energy representation, i.e.

$$\Omega^{(1)\ s}(s;s') = \sum_{\mu}\int \bar{\chi}^{(o)}(x,\mu;s)\Omega^{(1)x,\mu}\chi^{(o)}(x,\mu;s')\ dx \ .$$

Hence (80) becomes an integral equation for $u(v;s)$

$$(82) \quad \omega(v)u(v;s) = \omega(s)u(v;s) + c\int \Omega^{(1)s}(s';s)u(v;s')ds' \quad .$$

Equation (82) together with the orthogonality condition (70) determine $u(v;s)$, though not uniquely. The approach we have taken is, of course, quite conventional except, perhaps, for notation.

Friedrichs [5] has given sufficient conditions under which (82) has a solution. We shall derive some

results in a formal fashion.

We shall solve (82) by a perturbation procedure. Let us write

(83) $$u(v;s) = u^{(o)}(v;s) + e\,u^{(1)}(v;s) + e^{2}u^{(2)}(v;s) + \ldots \quad .$$

Substituting in (82) and (70) we have for the zero

(84) $$\omega(v)u^{(o)}(v;s) = \omega(s)u^{(o)}(v;s)$$

and

(85) $$\int \bar{u}^{(o)}(v;s)u^{(o)}(v;s')dv = \delta(s,s') \quad .$$

The solution of (84) and (85), aside from an unimportant phase factor, is

(86) $$u^{(o)}(v;s) = \delta(v;s) \quad .$$

The first order approximation is

(87) $$\omega(v)u^{(1)}(v;s) = \omega(s)u^{(1)}(v;s) + \Omega^{(1)s}(v;s) \quad .$$

The orthogonality condition gives

(88) $$u^{(1)}(v;s) + \overline{u^{(1)}}(s;v) = 0 \quad .$$

Solving for $u^{(1)}(v;s)$ with $v \neq s$ we have

(89) $$u^{(1)}(v;s) = \frac{\Omega^{(1)s}(v;s)}{\omega(v) - \omega(s)}$$

which satisfies condition (88).

We must now consider how to define $u^{(1)}(v;s)$ for v in the vicinity of s. This is best done by using symbolic functions in the sense of L. Schwartz [6]. Accordingly, we consider $u^{(1)}(v;s)$ to be

$$(90) \qquad u^{(1)}(v;s) = \lim_{\omega_0 \to 0} \frac{\Omega^{(1)}(v;s)}{\omega(v) - \omega(s) + i\,\omega_0}$$

or

$$(91) \quad u^{(1)}(v;s) = i\,\pi\,\Omega^{(1)}(v;s)\,\delta\big(\omega(v) - \omega(s)\big) + \frac{P\,\Omega^{(1)s}(v;s)}{\omega(v) - \omega(s)} \; .$$

The function $\delta\big(\omega(v) - \omega(s)\big)$ is defined in the second set of quantization variables s of page [3], where ω is one of the variables, in which case $\delta\big(\omega(v) - \omega(s)\big)$ is the ordinary Dirac δ-function. The operator $\frac{P\,\Omega^{(1)s}(v;s)}{\omega(v) - \omega(s)}$ means that the principal part of the integral involving this kernel is to be used, with the second set of quantization variables as the variables of integration.

An alternative solution for $u^{(1)}(v;s)$ is

$$u^{(1)}(v;s) = \lim_{\omega_0 \to 0} \frac{\Omega^{(1)s}(v;s)}{\omega(v) - \omega(s) - i\,\omega_0}$$

or

(92)

$$u^{(1)}(v;s) = -i\pi\,\Omega^{(1)s}(v;v)\,\delta\big(\omega(v) - \omega(s)\big) + \frac{P\,\Omega^{(1)s}(v;s)}{\omega(v) - \omega(s)} \; .$$

The functions $\pm i\pi\delta\left(\omega(v)-\omega(s)\right) + \frac{P}{\omega(v)-\omega(s)}$ are essentially the functions commonly called δ_+ and δ_- (see Møller [ ]).

The equation for $u^{(2)}(v;s)$ is similar to that for $u^{(1)}$. The solution again involved $\pm i\pi\delta\left(\omega(v)-\omega(s)\right)$ depending on whether one takes the limit of $\frac{1}{\omega(v)-\omega(s)+i\,\omega_0}$ or of $\frac{1}{\omega(v)-\omega(s)-i\,\omega_0}$ as $\omega_0 \to 0$.

The orthogonality condition leads to

$$(93) \quad \bar{u}^{(2)}(v;s) + u^{(2)}(s;v) = -\int \bar{u}^{(1)}(v;s)\, u^{(1)}(s;s')\, ds' \quad .$$

The solution of (82) is not unique. Friedrichs [5] discussed other possible solutions.

However, the two possible solutions which we have obtained are interesting because they are orthogonal and lead to the scattering operator. Without proof we assert that for a large class of operators $\Omega^{(1)}$ we have (see [5])

$$(94) \quad \begin{cases} \lim\limits_{t\to-\infty} e^{it\omega(s)} g(s)(t) = f(s) \\ \lim\limits_{t\to+\infty} e^{it\omega(s)} g(s)(t) = K\, f(s) \end{cases}$$

if the solution corresponding to (91) is used, K being the scattering operator. If the solution corresponding to (92)

is used one has

$$(95)\qquad \begin{cases} \lim\limits_{t\to-\infty} e^{it\omega(s)}g(s)(t) = K^{+}f(s) \\ \lim\limits_{t\to+\infty} e^{it\omega(s)}g(s)(t) = f(s) \end{cases}$$

where K^{+} is the scattering operator for time reversal. It should be noted that

$$(96)\qquad K^{+}K = I$$

Thus one should pick the solution $u(v;s)$ which satisfies the boundary condition one desires.

From (94) it is seen that for $t \to -\infty$ $g(s)(t)$ behaves like

$$(97)\qquad g(s)(t) = f(s)\, e^{-it\omega}$$

and hence is a solution of the Schrödinger equation with Hamiltonian $\Omega^{(o)}$. Likewise for $t \to +\infty$ $g(s)(t)$ is a solution of the Schrodinger equation with the Hamiltonian $\Omega^{(o)}$. In a sense one can say the perturbation has been "switched off" as $t \to \pm\infty$. Equations (95) have a similar interpretation. However, the "switching" on and off is not adiabatic because the states for $t \to \pm\infty$ differ, whereas in an adiabatic process they would be identical if the set of operators S consisted of $\Omega^{(o)}$ only (i.e., was non-degenerate).

The Canonical Transformation in the Electron-Positron Field.

We now wish to introduce the transformation which relates the representers in the M representation to those in the N representation. If we have the M- and N-representations for an arbitrary state vector Ψ, we write

$$\Psi \xleftrightarrow[M]{} \left\{ \zeta_n(v)_n \right\}$$

and

$$\Psi \longleftrightarrow \left\{ \psi_n(s)_n \right\} \quad .$$

We introduce the transformation in the form

$$(98) \quad \zeta_n(v)_n = \sum_{n'} \int T_{n,n'}(v)_{n'}:(s)_{n'} \; \psi_{n'}(s)_{n'}(ds)_{n'} \quad ,$$

where $(ds)_{n'}$ means an integration over the variables $s_1, s_2, \ldots, s_{n'}$. For convenience we use the notation $T_{n,n'}(v)_n:(s)_{n'}$ to be a function of the variables $v_1, v_2, \ldots, v_n$ and of the variables $s_1, s_2, \ldots, s_{n'}$. It is seen that this function is separately antisymmetric in $v_1, v_2, \ldots, v_n$ and $s_1, s_2, \ldots, s_{n'}$.

In order that the length of Ψ be preserved we have

$$\sum_n \int |\zeta_n(v)_n|^2 \, (dv)_n = \sum_{n'} \int |\psi_{n'}(s)_{n'}|^2 (ds)_{n'} \quad .$$

This equation leads to the orthogonality conditions for the functions $T(v)_n{:}(s)_{n'}$, namely

$$\tag{99} \left\{ \begin{aligned} &\sum_n \int \overline{T_{n;n'}}(v)_n{:}(s')_{n'}\, T_{n;n''}(v)_n{:}(s'')_{n''}(dv)_n \\ &\qquad = \delta_{n';n''} \underset{\substack{s'_1\ldots s'_{n'}\\ s''_1\ldots s''_{n''}}}{\mathrm{Asy}} \delta(s';s'')_{n'} \\ &\sum_{n'} \int \overline{T_{n;n'}}(v)_n{:}(s)_{n'}\, T_{n'';n'}(v'')_{n''}{:}(s)_{n'}(ds)_{n'} \\ &\qquad = \delta_{n;n''} \underset{\substack{v_1\ldots v_n\\ v''_1\ldots v''_{n''}}}{\mathrm{Asy}} \delta(v;v'')_n \end{aligned} \right.$$

where

$$\tag{100} \delta(s';s)_n = \delta(s'_1;s_1)\cdot\delta(s'_2;s_2)\ldots\delta(s'_n;s_n)$$

and

$$\delta(s';s)_o = 1 \quad .$$

We use the notation $\underset{\substack{v_1\ldots v_n\\ s_1\ldots s_{n'}}}{\mathrm{Asy}} f(v_1\ldots v_n;s_1\ldots s_{n'})$ to mean that $f(v_1\ldots v_n;s_1\ldots s_{n'})$ is to be antisymmetrized with respect to the two sets of variables $v_1\ldots v_n$ and $s_1\ldots s_{n'}$ separately.

It should be noted that the set of functions $T_{n;n'}(v)_n{:}(s)_{n'}$ is analogous to the function $u(v;s)$

used in the one-electron problem which transform functions in the s to the v representation. The whole point of what follows is to press the analogy with the one-electron problem to obtain the solution of the Schrödinger equation for the state function and other information derived therefrom. Though we shall not often mention the analogy in detail, it is this analogy which motivates much of what follows.

It will also be noted that the use of the number representation simplifies our problem considerably. If instead we had used an occupation number representation, the notation needed to keep the analogy clear would be so cumbersome as to hamper one seriously in the task of finding the transformation functions.

Before finding the functions, we shall indicate how they are used to solve the Schrödinger equation in the N-representation. As mentioned previously, the solution in terms of the M-representation is

$$(60) \qquad \zeta_n(v)_n(t) = e^{-it\left\{\sum_{i=1}^{n} |\omega(v_i)|\right\}} \zeta_n(v)_n$$

where $\zeta_n(v)_n$ is $\zeta_n(v)_n(0)$. Therefore, using (98) and (99), we have

$$(101) \qquad \psi_{n'}(s)_{n'}(t) = \sum_{n''} \int \psi_{n''}(s'')_{n''}(ds'')_{n''}$$

$$\sum_{n'} \quad \overline{T_{n;n'}}(v)_n:(s)_{n'} T_{n;n''}(v)_n:(s'')_{n''} \,\cdot$$

$$\cdot\; e^{-it\left\{\sum_{i=1}^{n} |\omega(v_i)|\right\}} (dv)_n$$

We shall now set up the equation for determining the functions $T_{n;n'}(v)_n:(s)_{n'}$.

Let us consider an arbitrary state vector Ψ with the M- and N-representations given by

$$\Psi \xleftrightarrow[M]{} \left\{\zeta_n(v)_n\right\}$$

$$\Psi \xleftrightarrow[N]{} \left\{\psi_n(s)_n\right\} \quad .$$

Then for any operator Λ we define operators $\Lambda^{v,n}$ and $\Lambda^{s,n}$ by

$$\Lambda\Psi \xleftrightarrow[M]{} \left\{\Lambda^{v;n} \zeta_n(v)_n\right\}$$

$$\Lambda\Psi \xleftrightarrow[N]{} \left\{\Lambda^{s;n} \psi_n(s)_n\right\} \quad .$$

That is to say, $\Lambda^{v,n}\zeta_n(v)_n$ is the n-particle component of $\Lambda\Psi$ in the M-representation and $\Lambda^{s,n}\psi_n(s)_n$ is the n-particle component of $\Lambda\Psi$ in the N-representation.

Now

$$(102) \; \Lambda^{v;n}\zeta_n(v)_n = \sum_{n'} \int T_{n;n'}(v)_n:(s)_{n'} \Lambda^{s;n'} \psi_{n'}(s)_{n'}(ds)_{n'} \quad .$$

If we have two state vectors $\Psi^{(1)}$ and Ψ and write the inner product $\left(\Psi^{(1)}, \Lambda\Psi\right)$ in terms of the N-representation we should have

$$\left(\Psi^{(1)}, \Lambda\,\Psi\right) = \sum_n \int \bar{\psi}_n^{(1)}(s)_n \,\Lambda^{s;n}\psi_n(s)_n(ds)_n \quad .$$

But since (by definition) the operator adjoint to Λ, which we denote by $\tilde{\Lambda}$, satisfies

$$\left(\tilde{\Lambda}\Psi^{(1)}, \ \Psi\right) = \left(\Psi^{(1)}, \Lambda\,\Psi\right)$$

we have

$$(103) \qquad \sum_n \int \bar{\psi}_n^{(1)}(s)\ \Lambda^{s;n}\psi_n(s)_n(ds)_n$$
$$= \sum_n \int \left[\overline{\tilde{\Lambda}^{s;n}\psi_n^{(1)}(s)_n}\right]\psi_n(s)_n(ds)_n \quad .$$

Hence we can write (102) as

$$(104) \quad \Lambda^{v;n}\zeta_n(v)_n = \sum_{n'} \int \left[\overline{\tilde{\Lambda}^{s;n'}}\,T_{n;n'}(v)_n{:}(s)_{n'}\right]\psi_{n'}(s)_{n'}(ds)_{n'}$$

where $\overline{\tilde{\Lambda}^{s;n'}}$ operates on the variables $s_1 \ldots s_{n'}$.

But we also have

$$(105)\ \Lambda^{v;n}\zeta_n(v)_n = \sum_{n'} \int \Lambda^{v;n}T_{n;n'}(v)_n{:}(s)_{n'}\psi_{n'}(s)_{n'}(ds)_{n'} \quad ,$$

where the operator $\Lambda^{v;n}$ operates on the variables $v_1 \ldots v_n$ of the functions $T_{n;n'}(v)_n{:}(s)_{n'}$.

Now since the state Ψ and hence the set of functions $\psi_n(s)_n$ is arbitrary we have from (104) and (105)

$$(106) \quad \Lambda^{v;n} T_{n;n'}(v)_n:(s)_{n'} = \overline{\tilde{\Lambda}^{s;n'}} T_{n;n'}(v)_n:(s)_{n'} \quad .$$

If we took Λ to be the energy operator of the field H, equation (106) for $T_{n;n'}(v)_n:(s)_{n'}$ would be analogous to the equations for $u(v;s)$. It is simpler, however, to take Λ to be one of the creation or annihilation operators $B^-(v)$, $B^+(v)$, $A^-(s)$, $A^+(s)$, all choices leading to the same result, as is required by a consistent theory. We take Λ to be the operator $B^-(v)$ and $\tilde{\Lambda}$ to be $B^+(v)$ since the creation operator is the adjoint of the annihilation operator.

The left-hand member of (106) is then

$$\sqrt{n+1} \quad T_{n+1;n'}(v)_n(v'):(s)_{n'}$$

The right-hand member of (106) is obtained in the following way. We can express the operator $B^+(v')$ in terms of $A^{\pm}(s)$ from (64) as

$$(107)\quad \overline{B^{+}}(v') =$$

$$= \overline{B^{+\iota'}}(v')\delta_{\iota',+1} + \overline{B^{-\iota'}}(v')\delta_{\iota',-1}$$

$$= \int u(v';s')\overline{A^{+\varepsilon'}}(s')ds'\ \delta_{\iota',+1} + \int \bar{u}(v';s')\overline{A^{-\varepsilon'}}(x')ds'\delta_{\iota',-1}$$

$$= \int [u_{++}(v';s') + \bar{u}_{--}(v';s')]\ \overline{A^{+}}(s')ds'$$

$$+ \int [u_{+-}(v';s') + \bar{u}_{-+}(v';s')]\ \overline{A^{-}}(s')ds'$$

where

$$(108)\quad \begin{cases} u_{++}(v';s') = u(v';s')\delta_{\iota',+1}\delta_{\varepsilon',+1} \\ u_{+-}(v';s') = u(v';s')\delta_{\iota',+1}\delta_{\varepsilon',-1} \end{cases}$$

etc.

Then we have

$$(109)\quad \overline{B^{+s;n'}}\ T_{n;n'}(v)_n:(s)_{n'}$$

$$= \sqrt{n'}\int [u_{++}(v';s) + \bar{u}_{--}(v';s)] \mathop{\mathrm{Asy}}_{s_1,s_2,\ldots,s_{n'}} T_{n;n'-1}(v)_n:(s)_{n'-1}$$

$$\cdot\ \delta(s_{n'};s)ds \quad .$$

$$+ \sqrt{n'+1}\int [u_{+-}(v';s) + \bar{u}_{-+}(v';s)]T_{n;n'+1}(v)_n:(s)_{n'}(s)ds \quad .$$

In (106) replacing n by $n-1$ and replacing v' by v_n, we are led to our fundamental equation for $T_{n;n'}(v)_n:(s)_{n'}$ which is

$$(110)\qquad \sqrt{n}\,T_{n;n'}(v)_n:(s)_{n'}$$

$$= \underset{s_1\ldots s_{n'}}{\mathrm{Asy}}\ \sqrt{n'}\,T_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1}[u_{++}(v_n;s_{n'}) + \bar{u}_{--}(v_n;s_{n'})]$$

$$+ \sqrt{n'+1}\int[u_{+-}(v_n;s') + \bar{u}_{-+}(v_n;s')]T_{n-1;n'+1}(v)_{n-1}:(s)_{n'}(s')ds'.$$

Equation (110) together with the orthogonality conditions (99) are our basic equations for finding the functions $T_{n;n'}(v)_n:(s)_{n'}$.

Equation (110) is a difference equation with respect to the subscripts n and an integral equation with respect to the variables s. We are thus led to an interesting integro-difference equation. To get some insight into the nature of the solution we shall consider a pure electron field, where we obtain a difference equation which can be solved explicitly.

In the case of the electron field where the M and N creation and annihilation operators are related by (64a), the equation which replaces (110) is

$$(110a)\quad \sqrt{n}\,T_{n;n'}(v)_n:(s)_{n'} =$$

$$= \sqrt{n'}\ \underset{s_1\ldots s_{n'}}{\mathrm{Asy}}\ T_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1}u(v_n;s_{n'}\quad .$$

The orthogonality conditions (99) are valid also for the electron field.

To solve (110a) it is convenient to think in terms of the n-n' plane.

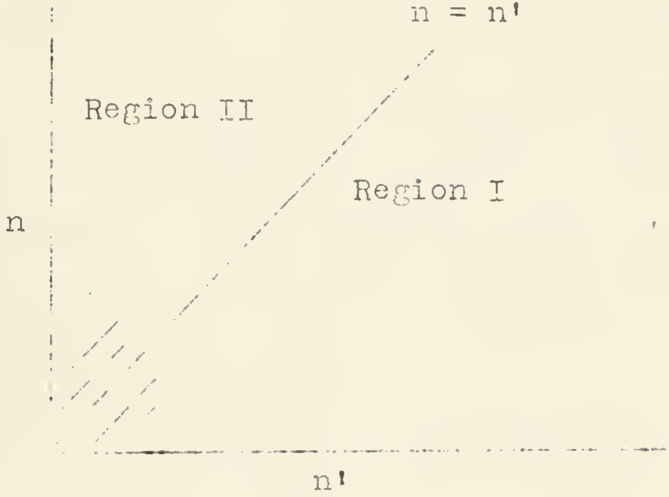

Figure 1

It is seen that if the functions $T_{n;n'}(v)_n:(s)_{n'}$ are prescribed on the n and n' axes, the functions can be found anywhere in the n-n' plane. It is further seen that the value of $T_{n;n'}(v)_n:(s)_{n'}$ at any point (n;n') depends only on the value of the function at the point on the axis at which a 45° line from (n;n') intersects the axis (see diagram). In Region II of the n-n' plane, i.e. the region for which $n > n'$, equation (110a) can be solved immediately, because the values of the functions $T_{n;n'}(v)_n:(s)_{n'}$ on the n-axis can be found

from (110a), except for the value $n = 0$. From (110a) it is clear that

$$T_{n;o}(v)_n:(s)_o = 0$$

so that

(111) $$T_{n;n'}(v)_n:(s)_{n'} = 0 \qquad \text{for} \qquad n > n' \quad .$$

In order to find the functions for $n' \geqq n$ we must know these functions on the n'-axis. For this we use the orthogonality conditions (99).

Setting $n' = n'' = 0$ in the first of the equations (99) and using (111) we have

(112) $$|T_{o;o}(v)_o:(s)_o|^2 = 1$$

or within an arbitrary but unimportant phase factor

(113) $$T_{o;o}(v)_o:(s)_o = 1 \quad .$$

Setting $n = n'' = 0$ in the second of equations (99) and using (112) we have

$$\sum_{n'>0} \int \overline{T_{o;n'}}(v)_o:(s)_{n'}\ T_{o;n'}(v)_o:(s)_{n'}\ (ds)_{n'} = 0$$

$$= \sum_{n'>0} \int |T_{o;n'}(v)_o:(s)_{n'}|^2 (ds)_{n'}$$

from which it follows that

$$(114) \qquad T_{0;n'}(v)_0:(s)_{n'} = 0 \qquad \text{for} \qquad n' > 0 \ .$$

Hence it follows that in Region I in the n-n' plane (i.e. for $n < n'$)

$$(115) \qquad T_{n;n'}(v)_n:(s)_{n'} = 0 \qquad\qquad n < n' \ .$$

Therefore it is only for $n' = n$ that transformation functions are not identically zero. From (110a) and (113) we have

$$(116) \quad T_{n;n}(v)_n:(s)_n = \underset{s_1 \ldots s_n}{\mathrm{Asy}} \; u(v_1;s_1)\, u(v_2;s_2) \ldots u(v_n;s_n)$$

$$= \underset{\substack{s_1 \ldots s_n \\ v_1 \ldots v_n}}{\mathrm{Asy}} \; \prod_{i=1}^{n} u(v_i;s_i)$$

where the second equation follows from the first by virtue of the properties of the asymmetrizing operator.

Thus the only transformation functions which do not vanish are these which take an n-particle state in the N-representation to an n-particle state in the M-representation. This result could have been anticipated because for the electron field

$$(117) \qquad\qquad M = N$$

as follows from (64a). It should also be noted that the

non-vanishing transformation function consists of appropriately symmetrized products of the single particle transformation functions. Again this fact could have been anticipated.

The greater complexity in the treatment of the electron-positron field arises from the fact that

$$M \neq N \tag{118}$$

in this case. This relation also accounts for the creation of electrons and positrons in the presence of an electromagnetic field, whereas in an electron field the number operator is a constant of the motion, so that the electromagnetic field does not create particles.

Solution of the Integro-Difference Equation.

We shall now solve equation (110) to the second order in the electron charge by a perturbation procedure.

In equation (110) we set up the expansion

$$(119)\quad T_{n;n'}(v)_n:(s)_{n'}$$

$$= T^{(0)}_{n;n'}(v)_n:(s)_{n'} + e\, T^{(1)}_{n;n'}(v)_n:(s)_{n'} + e^2\, T^{(2)}_{n;n'}(v)_n:(s)_{n'} + \ldots$$

$$(120)\quad u(v;s) = u^{(0)}(v;s) + e\, u^{(1)}(v;s) + e^2\, u^{(2)}(v;s) + \ldots\ .$$

Expansion (120) is identical with (83) so that the functions $u^{(0)}$, $u^{(1)}$ etc. are those calculated by the perturbation procedure discussed earlier. We shall consider as admissable functions $u(v;s)$ only those functions discussed earlier which are kernels of unitary transformations and lead to scattering operators for the single electron problem, since these functions guarantee that $B^{\pm}(v)$ are annihilation and creation operators.

With this restriction that only two functions $u(v;s)$ are to be used, we obtain from (110) as the zero-order equation using either,

(121) $$\sqrt{n}\, T^{(0)}_{n;n'}(v)_n:(s)_n$$

$$= \underset{s_1\ldots s_{n'}}{\mathrm{Asy}} \sqrt{n'}\, T^{(0)}_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1}\delta(v_n;s_{n'}) \quad .$$

The orthogonality conditions (99) give the results

(122) $$\sum_n \int T^{(0)}_{n;n'}(v)_n:(s')_{n'}\, T^{(0)}_{n;n''}(v)_n:(s'')_{n''}(dv)_n$$

$$= \delta_{n';n''} \underset{\substack{s_1\ldots s'_{n'} \\ s''_1\ldots s''_n}}{\mathrm{Asy}} \delta(s';s'')_{n'}$$

and

(123) $$\sum_{n'} \int \bar{T}^{(0)}_{n;n'}(v)_n:(s)_{n'}\, T^{(0)}_{n'';n'}(v'')_{n''}(s)_{n'}\,(ds)_{n'}$$

$$= \delta_{n;n'} \underset{\substack{v_1\ldots v_n \\ v''_1\ldots v''_n}}{\mathrm{Asy}} \delta(v;v'')_n \quad .$$

In a manner similar to the manner of solving for $T_{n;n'}(v)_n:(s)_{n'}$ in the case of the electron field we have for $T^{(0)}$

(124) $$T^{(0)}_{n,n'}(v)_n:(s)_{n'} = \delta_{n;n'} \underset{\substack{v_1\ldots v_n \\ s_1\ldots s_n}}{\mathrm{Asy}} \delta(v;s)_n \quad .$$

Thus the zero-order term gives an identity transformation, as is to be expected.

The first order terms from (110) are

$$\sqrt{n}\, T^{(1)}_{n;n'}(v)_n:(s)_{n'} = \underset{s_1\ldots s_{n'}}{\mathrm{Asy}} \sqrt{n'}\, T^{(1)}_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1}\delta(v_n;s_{n'})$$

$$+ \underset{s_1\ldots s_{n'}}{\mathrm{Asy}} \sqrt{n'}\, T^{(0)}_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1} \cdot$$

$$\cdot\, [u^{(1)}_{++}(v_n;s_{n'}) + \bar{u}^{(1)}_{--}(v_n;s_{n'})]$$

$$+\sqrt{n'+1}\int [u^{(1)}_{+-}(v_n;s') + \bar{u}^{(1)}_{-+}(v_n;s')] \quad \cdot$$

$$\cdot\, T^{(0)}_{n-1;n'+1}(v)_{n-1}:(s)_{n'}(s')ds'$$

or, substituting (124) for $T^{(0)}$,

$$(125)\quad \sqrt{n}\, T^{(1)}_{n;n'}(v)_n:(s)_{n'}$$

$$= \underset{s_1\ldots s_{n'}}{\mathrm{Asy}} \sqrt{n'}\, T^{(1)}_{n-1;n'-1}(v)_{n-1}:(s)_{n'-1}\delta(v_n;s_{n'})$$

$$+ \underset{s_1\ldots s_{n'}}{\mathrm{Asy}} \sqrt{n'}\, \delta_{n;n'}\delta(v;s)_{n'-1}[u^{(1)}_{++}(v_{n'};s_{n'}) + \bar{u}^{(1)}_{--}(v_{n'};s_{n'})]$$

$$+ \underset{\substack{v_1\ldots v_{n'+1}\\ s_1\ldots s_{n'}}}{\mathrm{Asy}} \sqrt{n'+1}\, \delta_{n;n'+2}\delta(v;s)_{n'}[u^{(1)}_{+-}(v_{n'+2};v_{n'+1})$$

$$+ \bar{u}^{(1)}_{-+}(v_{n'+2};v_{n'+1})] \quad .$$

The orthogonality relations also give a relation for $T^{(1)}$, namely,

$$(126)\qquad \overline{T^{(1)}_{n;n'}}(v)_n:(s)_{n'} + T^{(1)}_{n';n}(s)_{n'}:(v)_n = 0.$$

To solve (125) it is again useful to think in terms of the n-n' plane (see Figure 1). As in the case of the electron field, the value of $T^{(1)}_{n;n'}$ depends on the

values of this function on the n and n' axes. First let us consider the solution of (125) in Region II, where $n > n'$. We can obtain the values of $T^{(1)}_{n;n'}$ on the n-axis from (125) by setting $n' = 0$. It is seen that

$$(127) \quad T^{(1)}_{n;o}(v)_n:(s)_o = 0 \qquad n > 0 \ , \ n \neq 2 \ ,$$

$$T^{(1)}_{2;o}(v)_2:(s)_o = \frac{1}{\sqrt{2}}\,[u^{(1)}_{+-}(v_2;v_1) + \bar{u}^{(1)}_{-+}(v_2;v_1)] \ ,$$

$$= -\sqrt{2} \operatorname*{Asy}_{v_1,v_2} u^{(1)}_{+-}(v_1;v_2)$$

using the fact that $\bar{u}^{(1)}(v;s) = -u^{(1)}(s;v)$. Hence for $n > n'$ the only functions $T^{(1)}_{n;n'}(v)_n:(s)_{n'}$ which do not vanish are those for which $n = n'+2$.

The solution of (125) in Region II can be obtained by induction. The result is

$$(128) \quad T^{(1)}_{n+2;n}(v)_{n+2}:(s)_n$$

$$= -\sqrt{(n+1)(n+2)} \operatorname*{Asy}_{\substack{v_1\cdots v_{n+2}\\ s_1\cdots s_n}} \delta(v;s)_n \, u^{(1)}_{+-}(v_{n+1};v_{n+2}) \ .$$

Though the solution (128) is verified in a relatively simple way, it is perhaps instructive to carry out the verification, since it gives an example of how the asymmetrizing operator is used.

To verify (128) by induction we note that for $n = 0$ we have a correct solution. Let us now assume that

$T^{(1)}_{n+1;n-1}(v)_{n+1}:(s)_{n-1}$ is of the form (128) and we shall verify that $T^{(1)}_{n+2;n}(v)_{n+2}:(s)_n$ is also given by equation (128). Then by substituting $T_{n+1;n-1}(v)_{n+1}:(s)_{n-1}$ as given by (128) into (125) we have

$$T^{(1)}_{n+2;n}(v)_{n+2}:(s)_n$$

$$= -n\sqrt{\frac{n+1}{n+2}} \operatorname*{Asy}_{\substack{v_1\cdots v_{n+1}\\ s_1\cdots s_n}} \delta(v;s)_{n-1}u^{(1)}_{+-}(v_n;v_{n+1})\delta(v_{n+2};s_n)$$

$$- \sqrt{\frac{n+1}{n+2}} \operatorname*{Asy}_{\substack{v_1\cdots v_{n+1}\\ s_1\cdots s_n}} \delta(v;s)_n[u^{(1)}_{+-}(v_{n+1};v_{n+2}) - u^{(1)}_{+-}(v_{n+2};v_{n+1})]$$

or

$$(129)\quad T^{(1)}_{n+2;n}(v)_{n+2}:(s)_n$$

$$= -\sqrt{(n+1)(n+2)}\left\{\frac{n}{n+2} \operatorname*{Asy}_{\substack{v_1\cdots v_{n+1}\\ s_1\cdots s_n}} \delta(v;s)_{n-1}u^{(1)}_{+-}(v_n;v_{n+1})\delta(v_{n+2};s_n)\right.$$

$$\left. + \operatorname*{Asy}_{\substack{v_1\cdots v_{n+1}\\ s_1\cdots s_n}} \frac{1}{n+2}\,\delta(v;s)_n[u^{(1)}_{+-}(v_{n+1};v_{n+2}) - u^{(1)}_{+-}(v_{n+2};v_{n+1})]\right\}.$$

Now

$$\operatorname*{Asy}_{\substack{v_1\ldots v_{n+2}\\ s_1\ldots s_n}} \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+1};v_{n+2})$$

$$= \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \frac{1}{n+2}\, \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+1};v_{n+2})$$

$$- \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \frac{1}{n+2}\, \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+2};v_{n+1})$$

$$+ \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \frac{1}{n+2}\, \delta(v;s)_{n-1}\delta(v_{n+2};s_n)\, u^{(1)}_{+-}(v_n;v_{n+1})$$

$$- \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \frac{1}{n+2}\, \delta(v;s)_{n-2}\delta(v_{n+2};s_{n-1})\delta(v_{n-1};s_n)u^{(1)}_{+-}(v_n;v_{n+1})$$

$$+ \;\ldots\ldots$$

$$+ (-1)^{n-1} \operatorname*{Asy}_{\substack{v_1\ldots v_{n-1}\\ s_1\ldots s_n}} \frac{1}{n+2}\, \delta(v_{n+2};s_1)\delta(v_1;s_2)\;\ldots\;\delta(v_{n-1};s_n)\;\cdot$$

$$u^{(1)}_{+-}(v_n;v_{n+1}) \quad .$$

But the fourth and all higher terms of the sum are equal to the third term by virtue of the antisymmetrization in the variables $s_1\ldots s_n$. Hence

$$(130)\qquad \operatorname*{Asy}_{\substack{v_1\ldots v_{n+2}\\ s_1\ldots s_n}} \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+1};v_{n+2})$$

$$= \frac{1}{n+2} \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+1};v_{n+2})$$

$$- \frac{1}{n+2} \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \delta(v;s)_n\, u^{(1)}_{+-}(v_{n+2};v_{n+1})$$

$$+ \frac{n}{n+2} \operatorname*{Asy}_{\substack{v_1\ldots v_{n+1}\\ s_1\ldots s_n}} \delta(v;s)_{n-1}\delta(v_{n+2};s_n)\, u^{(1)}_{+-}(v_n;v_{n+1}) \;.$$

But the right-hand side of (130) is simply the expression in the curly brackets in (129). Hence we have verified that $T^{(1)}_{n+2;n}(v)_{n+2}:(s)_n$ has the solution (128).

Having found $T^{(1)}_{n;n'}(v)_n:(s)_{n'}$ in Region II of the n-n' plane where $n > n'$, we can easily find this function in Region I of the plane, i.e. for $n < n'$ by using the orthogonality relation (126). The result is

$$T^{(1)}_{n;n'}(v)_n:(s)_{n'} = 0 \qquad (n < n' \quad , \quad n \neq n'-2)$$

$$(131)\qquad T^{(1)}_{n-2;n}(v)_{n-2}:(s)_n = + \sqrt{n(n-1)} \operatorname*{Asy}_{\substack{v_1\ldots v_{n-2}\\ s_1\ldots s_n}} \delta(v;s)_{n-2}u^{(1)}_{-+}(s_{n-1};s_n) \;.$$

To find $T^{(1)}_{n;n}(v)_n:(s)_n$ we find $T^{(1)}_{o;o}(v)_o:(s)_o$ from (126) and $T^{(1)}_{n;n}(v)_n:(s)_n$ for $n > 0$ from (125). Thus we see that

(132) $$R\, T^{(1)}_{o;o}(v)_o:(s)_o = 0$$

where R stands for the real part of the function. Thus $T^{(1)}_{o;o}$ is purely imaginary. The phase of the representers can always be so chosen that the imaginary part contributes nothing. Thus

(133) $$T^{(1)}_{o;o}(v)_o:(s)_o = 0 \quad .$$

From (125) it can be proved by induction that

(134) $$T^{(1)}_{n;n}(v)_n:(s)_n$$

$$= n \underset{\substack{v_1 \cdots v_n \\ s_1 \cdots s_n}}{\mathrm{Asy}} \delta(v;s)_{n-1}[u^{(1)}_{++}(v_n;s_n) + \bar{u}^{(1)}_{--}(v_n;s_n)] \quad .$$

Note (134) includes (133).

We can summarize the results for $T^{(1)}_{n;n'}(v)_n:(s)_{n'}$ in the following way.

(135) $$T^{(1)}_{n;n'}(v)_n:(s)_{n'} = \delta_{n-2;n'} T^{(1)}_{n'+2;n'}(v)_{n'+2}:(s)_{n'}$$

$$+ \delta_{n;n'}\, T^{(1)}_{n';n'}(v)_{n'}:(s)_{n'}$$

$$+ \delta_{n+2;n'}\, T^{(1)}_{n'-2;n'}(v)_{n'-2}:(s)_{n'}$$

where $T^{(1)}_{n'+2;n'}$, $T^{(1)}_{n';n'}$, $T^{(1)}_{n'-2;n'}$ are given above.

In the same way the higher orders $T^{(p)}$ can be obtained. In fact, it appears possible to obtain a general expression for $T^{(p)}_{n;n'}(v)_n:(s)_{n'}$, though the expression would be complicated. We shall indicate the results for $T^{(2)}_{n;n'}(v)_n:(s)_{n'}$.

(136) $$T^{(2)}_{n;n'}(v)_n:(s)_{n'}$$

$$= \delta_{n-4;n'}T^{(2)}_{n'+4;n'}(v)_{n'+4}:(s)_{n'} + \delta_{n-2;n'}T^{(2)}_{n'+2;n'}(v)_{n'+2}:(s)_{n'}$$

$$+ \delta_{n;n'}T^{(2)}_{n';n'}(v)_{n'}:(s)_{n'} + \delta_{n+2;n'}T^{(2)}_{n'-2;n'}(v)_{n'-2}:(s)_{n'}$$

$$+ \delta_{n+4;n'}T^{(2)}_{n'-4;n'}(v)_{n'-4}:(s)_{n'}$$

where

(137) $$T^{(2)}_{n'+4;n'}(v)_{n'+4}:(s)_{n'}$$

$$= \frac{1}{2}\sqrt{(n'+1)(n'+2)(n'+3)(n'+4)} \operatorname*{Asy}_{\substack{v_1\cdots v_{n'+4}\\ s_1\cdots s_{n'}}} \delta(v;s)_{n'} \cdot$$

$$u^{(1)}_{+-}(v_{n'+1};v_{n'+2})\, u^{(1)}_{+-}(v_{n'+3};v_{n'+4})$$

(138) $T^{(2)}_{n'+2;n'}(v)_{n'+2}:(s)_{n'}$

$$= - \sqrt{(n'+1)(n'+2)} \underset{\substack{v_1 \dots v_{n'+2} \\ s_1 \dots s_{n'}}}{\text{Asy}} \Big\{ \delta(v;s)_{n'} \Big[u^{(2)}_{+-}(v_{n'+1};v_{n'+2})$$

$$+ \int u^{(1)}_{+-}(v_{n'+1};s') \, \bar{u}^{(1)}_{--}(v_{n'+2};s') ds' \Big]$$

$$+ n' \delta(v;s)_{n'-1} u^{(1)}_{+-}(v_{n'};v_{n'+1}) \Big[u^{(1)}_{++}(v_{n'+2};s_{n'}) + \bar{u}^{(1)}_{--}(v_{n'+2};s_{n'}) \Big] \Big\}$$

(139)

$$T^{(2)}_{n';n'}(v)_{n'}:(s)_{n'} = \underset{\substack{v_1 \dots v_{n'} \\ s_1 \dots s_{n'}}}{\text{Asy}} \Big\{ \delta(v;s)_{n'} \, T^{(2)}_{o;o}(v)_o:(s)_o$$

$$+ \frac{n'(n'-1)}{2} \delta(v;s)_{n'-2} \Big[u^{(1)}_{++}(v_{n'-1};s_{n'-1}) + \bar{u}^{(1)}_{--}(v_{n'-1};s_{n'-1}) \Big]$$

$$\cdot \Big[u^{(1)}_{++}(v_{n'};s_{n'}) + \bar{u}^{(1)}_{--}(v_{n'};s_{n'}) \Big]$$

$$- n'(n'-1) \, \delta(v;s)_{n'-2} \, u^{(1)}_{+-}(v_{n'-1};v_{n'}) \, u^{(1)}_{-+}(s_{n'-1};s_{n'})$$

$$+ n' \, \delta(v;s)_{n'-1} \Big[u^{(2)}_{++}(v_{n'};s_{n'}) + \bar{u}^{(2)}_{--}(v_{n'};s_{n'}) \Big]$$

$$+ n' \, \delta(v;s)_{n'-1} \int \Big[u^{(1)}_{+-}(v_{n'};s') \, \bar{u}^{(1)}_{+-}(s_{n'};s')$$

$$+ \bar{u}^{(1)}_{-+}(v_{n'};s') \, u^{(1)}_{-+}(s_{n'},s') \Big] \, ds' \Big\}$$

with

(140) $$T^{(2)}_{o;o}(v)_o:(s)_o = - \frac{1}{2} \iint \left| u^{(1)}_{+-}(s_1;s_2) \right|^2 ds_1 \, ds_2$$

$$(141)\quad T^{(2)}_{n'-2;n'}(v)_{n'-2}:(s)_{n'}$$

$$= \sqrt{n'(n'-1)} \underset{\substack{v_1\ldots v_{n'-2}\\ s_1\ldots s_{n'}}}{\mathrm{Asy}} \Big\{ \delta(v;s)_{n'-2}\Big[u^{(2)}_{-+}(s_{n'-1};s_{n'})$$

$$+ \int u^{(1)}_{--}(s_{n-1};s')\ \bar{u}^{(1)}_{+-}(s_n;s')ds'\Big]$$

$$+ (n'-2)\ \delta(v;s)_{n'-3}\Big[u^{(1)}_{++}(v_{n'-2};s_{n'-2}) + \overline{u^{(1)}_{--}}(v_{n'-2};s_{n'-2})\Big] \cdot$$

$$u^{(1)}_{-+}(s_{n'-1};s_{n'})\Big\}$$

$$(142)\quad T^{(2)}_{n'-4;n'}(v)_{n'-4}:(s)_{n'}$$

$$= -\frac{1}{2}\sqrt{(n'-3)(n'-2)(n'-1)n'} \underset{\substack{v_1\ldots v_{n'-4}\\ s_1\ldots s_{n'}}}{\mathrm{Asy}} \delta(v;s)_{n'-4} \cdot$$

$$u^{(1)}_{-+}(s_{n'-3};s_{n'-2})\ u^{(1)}_{-+}(s_{n'-1};s_{n'}) \quad .$$

There are several interesting results which should be particularly noted. Most important is that, while from the basic equation (110) it is by no means obvious that the functions $T_{n;n'}(v)_n:(s)_{n'}$ are antisymmetric in the variables $v_1\ldots v_n$, the solution has indeed these antisymmetry properties. This gratifying result is due to the facts that our choice of the functions

$u(v;s)$ guarantees that $B^{\pm}(v)$ satisfy the commutation rules for annihilation and creation operators when the operators $A^{\pm}(s)$ do, and also that we have used a consistent perturbation procedure.

Secondly we ought to observe that in the zero'th order $n = n'$ for non-vanishing transformation functions, whereas for first order $|n-n'| = 0,2$ and second order $|n-n'| = 0,2,4$ give non-vanishing functions. Generally, in the p'th order non-vanishing transformation functions occur for $|n-n'| = 0,2,4,\ldots,2p$. From an analysis to be considered later it can be shown that this result corresponds to the possibility of creating $0,1,2,\ldots,p$ electron-positron pairs.

Another important consequence of the way the theory is set up is that the theory is *á-priori* guage invariant up to this point. This result is an immediate consequence of the guage invariance of the Dirac Hamiltonian. That is, the spectrum of the Dirac Hamiltonian is unchanged when the guage of the electromagnetic field is changed, so that the transformation functions $u(v;s)$ introduced earlier are unaffected by such a change in guage.

The Time-Variation of Expectation Values.

From (101) we are now in a position to find how the N-representers $\psi_n(s)_n(t)$ vary in time, at least to the second order in the electron charge.

We shall first consider the problem of the existence of the scattering operator. From single particle electron theory, the scattering operator relates the functions

$$\lim_{t\to-\infty} e^{it\omega(s)}\psi(s)(t) =$$

$$= \lim_{t\to-\infty} \int \bar{u}(v;s)\zeta(v)e^{-it[\omega(v)-\omega(s)]}dv = \zeta(s)$$

and the function

$$\lim_{t\to+\infty} e^{it\omega(s)}\psi(s)(t)$$

$$= \lim_{t\to+\infty} \int \bar{u}(v;s)\zeta(v)e^{-it[\omega(v)-\omega(s)]}dv = f(s)$$

i.e. the scattering operator is given by the kernel $K(s;s')$ which relates $f(s)$ and $\zeta(s)$ in the following way

$$f(s) = \int K(s;s')\,\zeta(s')\,ds \quad . \tag{143}$$

(See Friedrichs [5] for details).

In order for the scattering operator exist it is necessary that $\lim_{t \to \pm\infty} e^{i\omega(s)}\psi(s)(t)$ exist.

We shall employ an analogous definition for the scattering operator in field theory. We say that the scattering operator relates the functions

$$\lim_{t \to -\infty} e^{it\left[\sum_{i=1}^{n} |\omega(s_i)|\right]} \psi_n(s)_n(t)$$

to the functions

$$\lim_{t \to +\infty} e^{it\left[\sum_{i=1}^{n} |\omega(s_i)|\right]} \psi_n(s)_n(t) \quad .$$

However, in the case of the electron-positron field such limits do not exist. This result is a consequence of the fact that the functions $T_{n;n'}(v)_n:(s)_{n'}$ do not vanish identically for $n \neq n'$.

Consider

$$e^{it\left[\sum_{i=1}^{n} |\omega(s_i)|\right]} \psi_n(s)_n(t)$$

$$= \sum_{n'} \int (dv)_{n'} \overline{T}_{n';n}(v)_{n'}:(s)_n \zeta_{n'}(v)_{n'} e^{it\left[\sum_{i=1}^{n}|\omega(s_i)| - \sum_{i=1}^{n'}|\omega(v_i)|\right]} .$$

Then, because n can differ from n', it is clear that the exponent on the right-hand side of the equation will

contain oscillating factors for $n \neq n'$ which cannot approach a limit as $|t| \to \infty$. For the case of an electron field, however, since $T_{n;n'}(v)_n:(s)_{n'}$ vanishes for $n \neq n'$, the limit will exist if $T_{n;n}(v)_n:(s)_n$ is a sufficiently smooth function of its arguments, and the scattering operator can be constructed out of suitably symmetrized one particle scattering operators,

Returning to the electron-positron field we see that as a consequence of the above considerations, no scattering operator exists in the sense of a scattering operator in single particle theory.

This result is entirely analogous to the situation in single particle theory where the zero-order Hamiltonian has point eigenvalues and the perturbation is such that transitions are possible between the point eigenvalues.

However, there is a sense in which the scattering operator exists, i.e. when fields $\phi(x)$, $A_i(x)$ are switched on adiabatically. If we prescribe an arbitrary state Ψ in the N representation at time $t \to -\infty$ as

$$\Psi \xleftrightarrow[N]{} \left\{\psi_n(s)_n\right\}$$

then at any finite time t one has

$$(144) \qquad \Psi(t) \underset{M}{\longleftrightarrow} \left\{ \psi_n(v)_n \, e^{-it \sum_{i=1}^{n} |\omega(v_i)|} \right\}$$

where the functions $\psi_n(v)_n$ are the same as those used to describe Ψ in the N representation at time $t \to -\infty$.

Hence if we represent $\Psi(t)$ in the N representation as

$$\Psi(t) \underset{N}{\longleftrightarrow} \left\{ \psi_n(s)_n(t) \right\}$$

we have

$$(145) \qquad \psi_n(s)_n(t) = \sum_{n'} \overline{T_{n';n}}(v)_{n'}(s)_n \psi_{n'}(v)_{n'} e^{-it\sum_{i=1}^{n'} |\omega(v_i)|} (dv)_{n'}$$

which relates the N-representers at time t to those at time $t \to -\infty$. Hence one may regard equation (145) as giving $\psi_n(s)_n(t)$ in terms of $\psi_n(s)_n(t \to -\infty)$ through the use of a scattering operator which is given by the functions $T_{n;n'}(v)_n(s)_{n'}$ which we have calculated above.

Strictly speaking we should prove assertion (144). However, results of this type seem to be generally

true in quantum theory. (See e.g. Born [8] and Snyder [9] for single particle theory and Friedrichs [10] for an example in the theory of quantum fields.)

We shall now calculate the expectation values of various operators. One can calculate these expectation values in two different ways corresponding to two types of boundary conditions which can be used.

The first boundary condition consists of considering the electromagnetic field to be constant in time and to prescribe the state $\Psi(0)$ in terms of its N-representers and then find the expectation values of the operators for any time t, in particular for $|t| \to \infty$ by considering $\Psi(t)$ in terms of its N-representers at time $t = 0$, as given by equation (101).

The other boundary condition consists of considering the fields to be switched on adiabatically and prescribing $\Psi(t \to -\infty)$ in terms of its N-representers and using equation (145) to express $\Psi(t)$ in terms of the N-representers.

The second boundary condition is the one most often used by previous workers, but we shall indicate also results of both types of boundary conditions. For

the most part we shall consider vacuum expectations, i.e. where $\Psi(0)$ or $\Psi(t \to -\infty)$ is taken to contain zero particles, according as the first or second boundary condition is used.

Let us now consider the expectation value of the number operator N using the first boundary conditions. For this case only we shall consider $\Psi(0)$ to be perfectly general in order to show the nature of the interference of various particle states in the expectation values of any operator. Then representing $\Psi(0)$ by

$$\Psi(0) \underset{N}{\longleftrightarrow} \left\{ \psi_n(s)_n \right\}$$

one can show to second order by a straightforward, though tedious, calculation

$$(146) \qquad \Big(\Psi(t), N\,\Psi(t)\Big) = \Big(\Psi(0), N\,\Psi(0)\Big)$$

$$- 4eR \sum_{n=0}^{\infty} \sqrt{n(n-1)} \int \overline{\psi}_n(s)_n \psi_{n-2}(s)_{n-2}\, u^{(1)}_{+-}(s_{n-1};s_n) \Big[1 - e^{it\left[|\omega(s_{n-1})| + |\omega(s_n)|\right]}\Big] (ds)_n$$

$$- 4e^2 \iint \left| u^{(1)}_{+-}(s;s') \right|^2 \Big[\cos t \Big(|\omega(s')| + |\omega(s'')| \Big) - 1 \Big] \, ds' ds''$$

$$+ 2e^2 \sum_{n=0}^{\infty} \iiiint \overline{\psi}_n(s)_{n-1}(s') \psi_n(s)_{n-1}(s'') \Big[u^{(1)}_{+-}(s';\overline{s}) \overline{u}^{(1)}_{+-}(s'';\overline{s}) +$$

$$\overline{u}^{(1)}_{-+}(s';\overline{s}) u^{(1)}_{-+}(s'';\overline{s}) \Big] \Big[e^{-it\left[|\omega(s'')| + |\omega(\overline{s})|\right]} -$$

$$e^{it\left[|\omega(s')| - |\omega(s'')|\right]} + e^{it\left[|\omega(s')| + |\omega(\overline{s})|\right]} - 1 \Big] ds' ds'' d\overline{s} (ds)_{n-1}$$

$$+ 4e^2 \, R \sum_{n=0}^{\infty} \sqrt{n(n-1)(n-2)(n-3)} \int \overline{\psi}_n(s)_n \psi_{n-4}(s)_{n-4} u^{(1)}_{+-}(s_{n-3};s_{n-2}) \cdot$$

$$u^{(1)}_{+-}(s_{n-1};s_n) \Big[e^{it\left[|\omega(s_{n-1})| + |\omega(s_n)|\right]} -$$

$$e^{it\left[|\omega(s_{n-3})| + |\omega(s_{n-2})|\right]} \Big] (ds)_n$$

$$- 4e^2 \, R \sum_{n=0}^{\infty} \sqrt{n(n-1)} \int \overline{\psi}_n(s)_n \, \psi_{n-2}(s)_{n-2} \Big\{ \Big[1 -$$

$$e^{it\left[|\omega(s_{n-1})| + |\omega(s_n)|\right]} \Big] u^{(2)}_{+-}(s_{n-1};s_n) +$$

$$\int d\overline{s} \Big[e^{-it\left[|\omega(s_{n-1})| + |\omega(s_n)|\right]} -$$

$$e^{it\left[|\omega(s)_n| + |\omega(\overline{s})|\right]} \Big] u^{(1)}_{--}(s_{n-1};\overline{s}) \, u^{(1)}_{+-}(s_n;\overline{s}) +$$

$$\int d\overline{s} \Big[1 - e^{it\left[|\omega(s_n)| + |\omega(\overline{s})|\right]} \Big] u^{(1)}_{++}(s_{n-1};\overline{s}) \, u^{(1)}_{-+}(s_n;\overline{s}) \Big\} (ds)_n$$

Here R stands for the real part. It should be noted that for t sufficiently small, the expectation value is proportional to t. Further, if $\Psi(0)$ corresponds to exactly n particles, then the first order term vanishes so that the expectation value of number of particles is constant to first order.

For $|t| \to \infty$, the above expression simplifies if one assumes the functions $u(v;s)$ are smooth enough so that one can apply the Riemann-Lebesgue theorem. One can always make this assumption by assuming a very smooth interaction in the Dirac Hamiltonian, if necessary even more general than that corresponding to interaction with the point electron. After the expression for $|t| \to \infty$ is obtained, one can then consider the limiting case when the interaction becomes that for the given electromagnetic field with point electron. We shall always think in terms of such a limiting process to enable us to use the Riemann-Lebesgue theorem. Oftentimes physicists, instead of using such concepts, prefer to say that they take the average value in time of the oscillating terms. However, our points of view is perhaps more consistent with the attitude used in the classical theory of the electron.

One has then

$$(147) \quad \lim_{|t| \to \infty} \Big(\Psi(t), N \Psi(t) \Big)$$

$$= \Big(\Psi(0), N \Psi(0) \Big) - 4c\, R \sum_{n=0}^{\infty} \sqrt{n(n-1)} \int \overline{\psi}_n(s)_n \, \psi_{n-2}(s)_{n-2} \cdot$$

$$u^{(1)}_{+-}(s_{n-1};s_n)(ds)_n + c^2 \Big\{ 4 \iint |u^{(1)}_{+-}(s';s'')|^2 ds' \, ds''$$

$$- 2 \sum_{n=0}^{\infty} n \int (ds)_n \Big[\Big| \int \psi_n(s)_{n-1}(s') \, \overline{u}^{(1)}_{+-}(s';s_n) ds' \Big|^2$$

$$+ \Big| \int \psi_n(s)_{n-1}(s') \, u^{(1)}_{-+}(s';s_n) ds' \Big|^2 \Big]$$

$$- 4R \sum_{n=0}^{\infty} \sqrt{n(n-1)} \int \overline{\psi}_n(s)_n \, \psi_{n-2}(s)_{n-2} \Big[u^{(2)}_{+-}(s_{n-1};s_n)$$

$$+ \int u^{(1)}_{++}(s_{n-1};s') \, u^{(1)}_{+-}(s_n;s') ds' \Big] (ds)_n \Big\} .$$

If $\Psi(0)$ contains zero particles we obtain as the limit of the vacuum expectation of the number operator

(148)

$$\lim_{|t| \to \infty} \Big(\Psi(t), N \Psi(t) \Big) = 4c^2 \iint \Big| u^{(1)}_{+-}(s';s'') \Big|^2 ds' \, ds'' \quad .$$

The expression in (148) is essentially that for $T^{(2)}_{o;o}(v)_o(s)_o$ (see equation (140)) whose convergence will be investigated in Part II.

Considering now the second boundary condition or the case of adiabatic switching on of the field for

which the state at $t \to -\infty$ consists of zero particles we obtain for any time t, the vacuum expectation

$$(148(\quad \left(\Psi(t), N\Psi(t)\right) = 2e^2 \iint \left| u^{(1)}_{+-}(s';s'') \right|^2 ds'\, ds''$$

which is just $\frac{1}{2}$ of the result using the first boundary condition. The fact that the first order term in e is zero is often expressed by saying that one has "virtual pair-production to the first order". This is the case usually considered in the literature when time-dependent potentials are discussed and more general effects are possible, instead of time-independent potentials, as in the present paper.

One can also calculate the expectation value of the energy operator $H^{(o)}$ corresponding to the kinetic energy of the electron-positron field. For the case of the first boundary condition taking $\Psi(0)$ to consist of zero particles we find

$$(149) \quad \left(\Psi(t), H^{(o)}\Psi(t)\right)$$

$$= 2e^2 \iint \left| u^{(1)}_{+-}(s';s'') \right|^2 \left[1 - \cos t\left(|\omega(s')| + |\omega(s'')|\right)\right] \cdot \left[|\omega(s')| + |\omega(s'')|\right] ds'\, ds''$$

or

$$\lim_{|t|\to\infty} \left(\Psi(t), H^{(o)}\Psi(t)\right)$$

$$= 2e^2 \iint \left|u^{(1)}_{+-}(s';s'')\right|^2 \left[|\omega(s')| + |\omega(s'')|\right] ds'\, ds'' \quad .$$

For the adiabatic case, if one starts out with zero particles at $t \to -\infty$, one finds

$$(150) \quad \left(\Psi(t), H^{(o)}\Psi(t)\right)$$

$$= \iint \left|u^{(1)}_{+-}(s';s'')\right|^2 \left[|\omega(s')| + |\omega(s'')|\right] ds'\, ds''$$

which is again $\frac{1}{2}$ the previous result. Friedrichs [10] has obtained similar results involving factors of $\frac{1}{2}$ in his study of the meson field.

We shall now consider the expectation value of the charge operator and the charge density operator.

If we take the first boundary condition, discussed earlier, in which the applied electromagnetic fields are constant in time and we specify $\Psi(0)$ in the Schrödinger picture to be a 0-particle state, it is clear from the constancy of the charge operator in the Heisenberg picture that

$$(151) \qquad \left(\Psi(t), Q\Psi(t)\right) \equiv 0$$

for all time.

If we take the second boundary condition in which the fields are switched on adiabatically, and take $\Psi(-\infty)$ to be a zero particle state, we have that $\Psi(t)$ is a zero particle state in the M-representation. Now (as is easily shown) Q has the same expression in terms of the operators $B^{\pm}(v)$ as it has in terms of $A^{\pm}(s)$ we have again

$$(\Psi(t), Q\Psi(t)) \equiv 0. \tag{152}$$

Thus with either boundary condition we have that the expectation value of the charge is zero for finite time.

We can calculate the expectation value $(\Psi(t), \rho(x)\Psi(t))$ using either of the above boundary conditions. It will turn out that this expectation value does not vanish identically for finite x. However, we know that $\int dx\, (\Psi(t), \rho(x)\Psi(t)) = (\Psi(t), Q\Psi(t)) = 0$, which means that the integrated charge density must vanish. We interpret this result by saying that one has positron-electron pairs created by the presence of an electromagnetic field. This situation is analogous to the classical case of the polarization of a dielectric by an

external electromagnetic field. In quantum electrodynamics one calls the expectation value of $\rho(x)$, $j_i(x)$ the "polarization of the vacuum" if the state corresponds to the presence of zero particles initially.

Using the first boundary condition one can calculate $\left(\Psi(t), \rho(x)\Psi(t)\right)$ in a straightforward way, using (41). One finds that the zero order vanishes. The result to the first order is

$$(153) \qquad \left(\Psi(t), \frac{\rho(x)}{e}\Psi(t)\right)$$

$$= 2eR \sum_{\mu} \iint u^{(1)}_{-+}(s_1;s_2)\Big(1 - e^{it\left[|\omega(s_1)| + |\omega(s_2)|\right]} \quad .$$

$$\chi^{(0)}(x,\mu;s_1)\, \bar{\chi}^{(0)}(x,\mu;s_2)\, ds_1\, ds_2 \quad .$$

R stands for the real part. We can easily show that as required

$$\int dx \left(\Psi(t), \frac{\rho(x)}{e}\Psi(t)\right) \equiv 0 \quad ,$$

for from the orthogonality conditions on $\chi^{(0)}$

$$\sum_{\mu} \int dx\, \bar{\chi}^{(0)}(x,\mu;s_2)\, \chi^{(0)}(x,\mu;s_1) = \delta(s_2 - s_1)$$

but because $u^{(1)}_{-+}(s_1;s_2)$ vanishes for $s_1 = s_2$ we have

our desired result immediately.

We have as $|t| \to \infty$

$$(154) \qquad \lim_{|t| \to \infty} \left(\Psi(t), \frac{\rho(x)}{e} \Psi(t) \right)$$

$$= 2eR \sum_{\mu} \iint u^{(1)}_{-+}(s_1;s_2)\, \chi^{(0)}(x,\mu;s_1)\, \bar{\chi}^{(0)}(x,\mu;s_2)\, ds_1\, ds_2$$

The integral is seen to be essentially the χ-representation of the kernel $u^{(1)}_{-+}(s_1;s_2)$. Further,

$$\lim_{|t| \to \infty} \int \Psi(t), \frac{\rho(x)}{e} \Psi(t) dx = \lim_{|t| \to \infty} \left(\Psi(t), Q \Psi(t) \right) = 0$$

as before.

The second boundary condition with adiabatic switch-on of the external field results again in the expression (154).

Similar results are obtained for the current density operator except now one does not have conservation of the total current, the reason being that the current operator, or equivalently the total momentum, is not a constant of the motion. Thus with the first or second boundary condition one has

$$(155) \qquad \lim_{|t| \to \infty} \left(\Psi(t), \frac{j_i(k)}{e} \Psi(t) \right)$$

$$= 2eR \sum_{\mu} \int u^{(1)}_{-+}(s_1;s_2)\, \chi^{(0)}(x,\mu;s_1)\, \bar{\alpha}^{\mu}_{i}\, \bar{\chi}^{(0)}(x,\mu;s_2)\, ds_1\, ds_2 \quad .$$

Acknowledgment

The author wishes to express his indebtedness to Professor K. O. Friedrichs for numerous helpful discussions.

REFERENCES

[1.] Friedrichs, K. O., Mathematical aspects of quantum theory of fields, Parts I and II, Communications on Pure and Applied Mathematics, Volume IV, 1951, pp. 1-63.

[2.] Yang, C. N., and Feldman, D., The S-matrix in the Heisenberg representation, Physical Review, Volume 79, 1950, pp. 972-978.

[3.] Kroll, N., Lectures on quantum electrodynamics, (1949-1950), Sections 7 and 8, Columbia University.

[4.] Schwinger, J., Quantum electrodynamics I, A covariant formulation, Physical Review, Volume 74, 1948, pp. 1439-1461.

[5.] Friedrichs, K. O., On the perturbation of continuous spectra, Communications on Pure and Applied Mathematics, Volume I., 1948, pp. 361-406.

[6.] Schwartz, L., Theorie des distributions, Hermann et Cie., Paris, 1950, 1951.

[7.] Møller, C., General properties of the characteristic matrix in the theory of elementary particles I. Det. Kgl. Danske Videnskolernes Selskob. Matematisk-Fysiske Meddeleser, Bind XXIII, 1945, pp. 1-48.

[8.] Born, M., Das adiabatatenprinzip in der quantenmechanik, Zeitschrift für Physik, Volume 40, 1926, pp. 167-192.

[9.] Snyder, H. S., Remarks concerning the adiabatic theorem and the S-matrix, Physical Review, Volume 83, 1951, pp. 1154-1159.

[10.] Friedrichs, K. O., Mathematical aspects of the quantum theory of fields, Part VII, Communications on Pure and Applied Mathematics, Volume V, 1952, pp. 1-56.

Date Due

JUN 5 - 1966			
MAY 17 1972			

Demco 38-297

IMM-NYU 179

IMM-NYU 179

AUTHOR
Moses, H. E.

TITLE Electrons and Positrons

BORROWER'S NAME ROOM NUMBER

Moses, H. E.
IMM-NYU 179.
Electrons and positrons

~~NEW YORK UNIVERSITY~~

~~INSTITUTE [illegible] SCIENCES~~

25 ~~Waverly Place, New York 3, N. Y.~~